Journey through Mechanics Research: A Personal Retrospective

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City U Distinguished Lecture

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CONTENTS OF MY LECTURE

- **Professional retrospectives:**
  - **Primal-dual variational principles (PhD)**
  - **Hypervelocity impact (Lockheed)**
  - **Modeling of geological phenomena (OU)**
  - **Modeling of bimodular materials (OU)**
  - **Third-order structural theories (VPI)**
  - **Penalty finite element models of flows of viscous incompressible fluids (VPI)**
  - **Layerwise laminate theory (VPI & TAMU)**
  - **Robust shell element (TAMU)**
  - **Modeling of biological cells (TAMU)**
  - **Least-squares FE models of fluid flow (TAMU)**
  - **Strain gradient, non-local, and non-classical continuum models (TAMU)**
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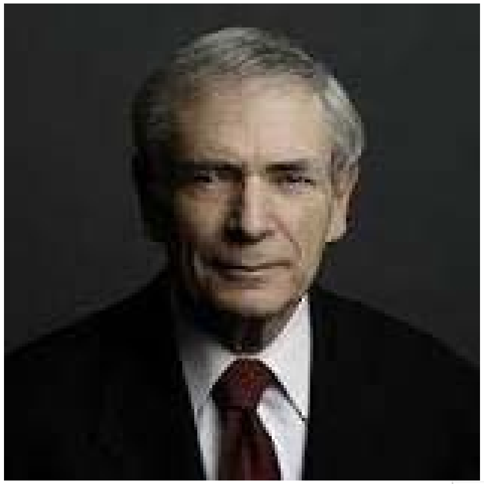
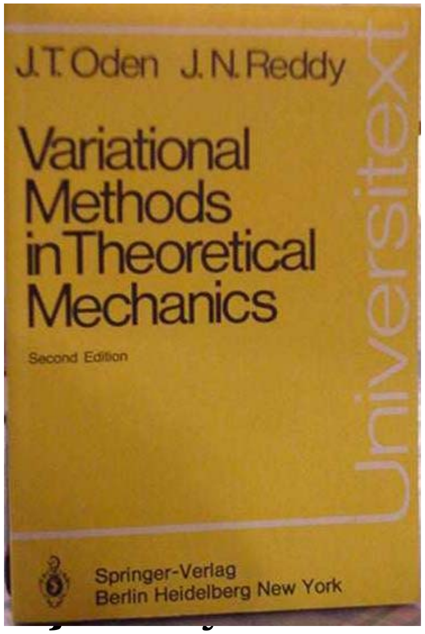
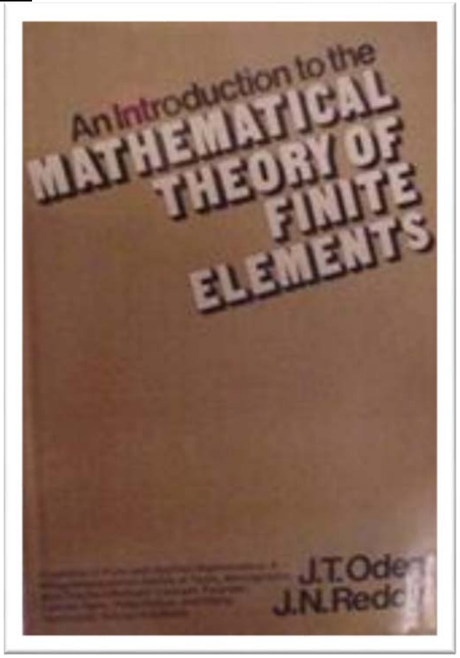
- **Closing remarks**

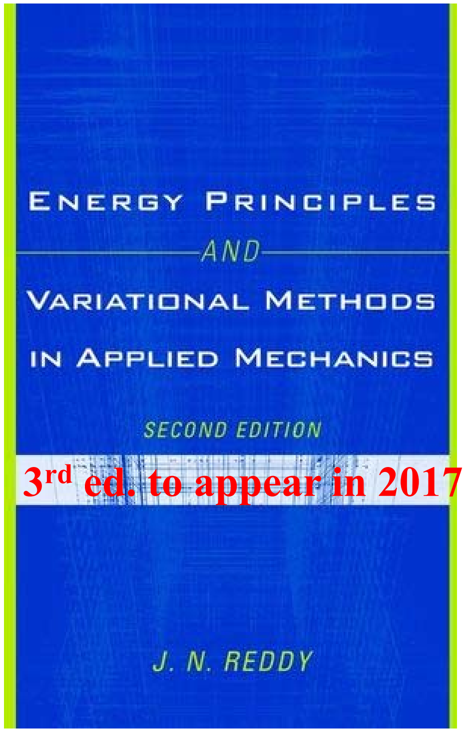
ON PRIMAL-DUAL VARIATIONAL PRINCIPLES IN MECHANICS

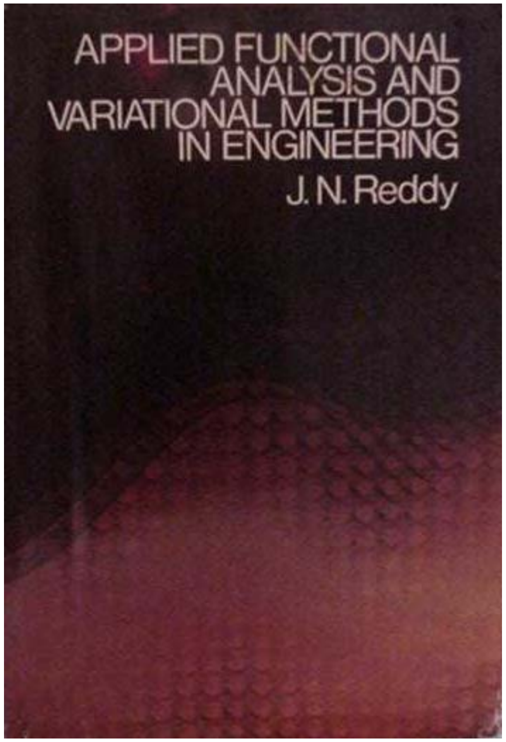


Professor J. Tinsley Oden, Director, Institute for Computational Engineering and Sciences; Professor of Aerospace Engineering and Engineering Mechanics, University of Texas at Austin (**JN's Ph.D. Thesis advisor and coauthor of papers and books**).









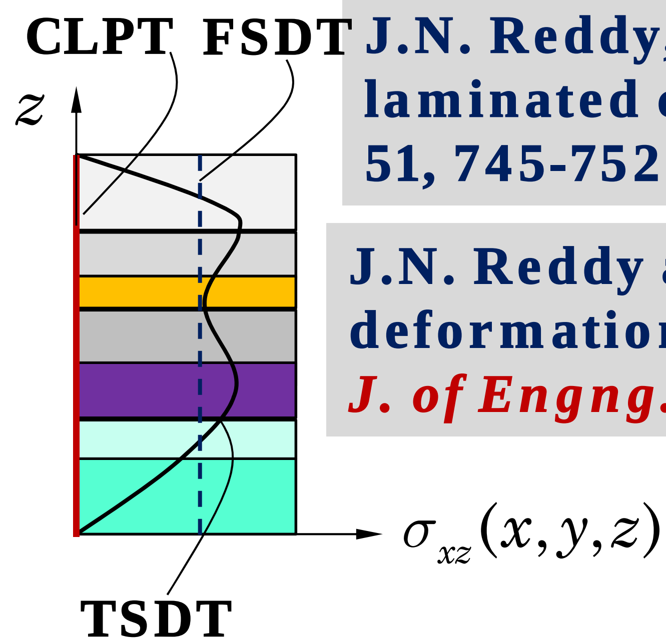
ON PRIMAL-DUAL VARIATIONAL PRINCIPLES IN MECHANICS

J.T. Oden and J.N. Reddy, "On dual-complementary variational principles in mathematical physics," *Int. J. Engng Science*, 12, 1-29 (1974). Supported by AFOSR

- Variational principles for $T^*ET(u) + f = 0$ in Ω and
  $S^*CS(\chi) + \eta = 0$ in Ω
- 14 Variational principles of elasticity: 7 primal and 7 dual;
- Fluid mechanics, electrostatics, magnetostatics; and nonlinear operators

All conventional as well as mixed variational principles are derived. Several of these principles formed the basis of the mixed, hybrid, and assumed strain finite element models (they were not cited often because our work was a bit mathematical and buried in the literature).

THIRD-ORDER SHEAR DEFORMATION THEORIES



J.N. Reddy, "A simple higher-order theory for laminated composite plates," ***J. of Applied Mechanics*****, 51, 745-752 (1984).** **(over 2000 citations)**

J.N. Reddy and C.F. Liu, "A higher-order shear deformation theory for laminated elastic shells," ***Int. J. of Engng. Sci.*****, 23(3), 319-330 (1985).** **(800 citations)**

Transverse shear stress

$$\sigma_{xz}(x,y,z) = Q_{55}\gamma_{xz} + Q_{45}\gamma_{yz}$$

$$\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}; \quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$

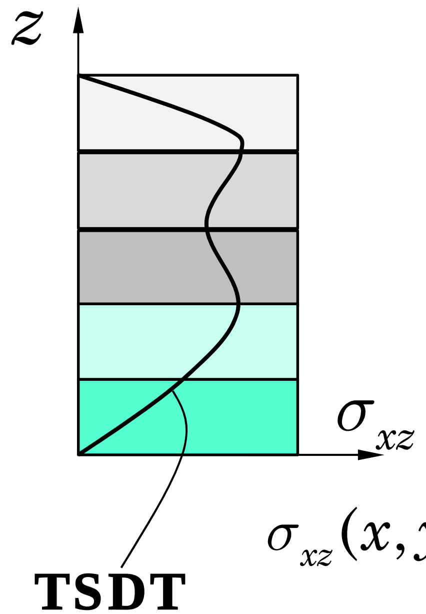
Displacement field

$$u(x,y,z) = u_0(x,y) + z\phi_x(x,y) + z^2\theta_x(x,y) + z^3\lambda_x(x,y)$$

$$v(x,y,z) = v_0(x,y) + z\phi_y(x,y) + z^2\theta_y(x,y) + z^3\lambda_y(x,y)$$

$$w(x,y,z) = w_0(x,y)$$

REDDY'S THIRD-ORDER SHEAR DEFORMATION THEORY



Transverse shear strains

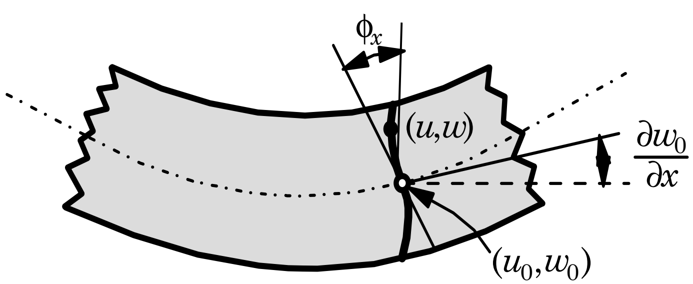
$$\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \phi_x(x,y) + 2z\theta_x(x,y) + 3z^2\lambda_x(x,y) + \frac{\partial w}{\partial x}$$

$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = \phi_y(x,y) + 2z\theta_y(x,y) + 3z^2\lambda_y(x,y) + \frac{\partial w}{\partial y}$$

Vanishing of transverse shear stresses on the bounding planes

$$\sigma_{xz}(x,y,\pm h/2) = \sigma_{yz}(x,y,\pm h/2) = 0 \Rightarrow \theta_x(x,y) = \theta_y(x,y) = 0$$

$$\lambda_x(x,y) = -\frac{4}{3h^2}\left[\phi_x(x,y) + \frac{\partial w}{\partial x}\right],\ \lambda_y(x,y) = -\frac{4}{3h^2}\left[\phi_y(x,y) + \frac{\partial w}{\partial y}\right]$$



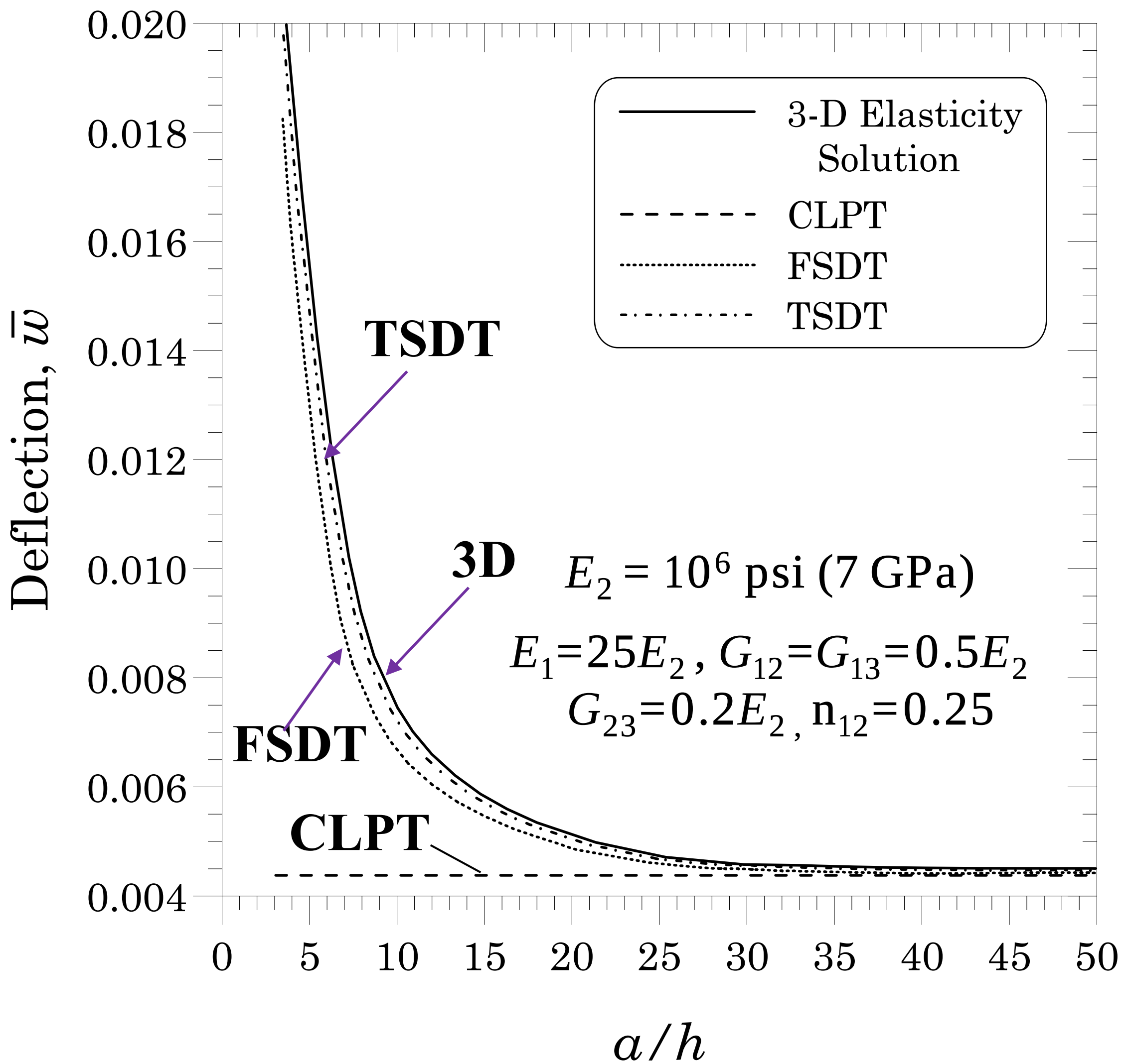
JN Reddy

$$u(x,y,z) = u_0(x,y) + z\phi_x(x,y) - \frac{4z^3}{3h^2}\left(\phi_x + \frac{\partial w}{\partial x}\right)$$

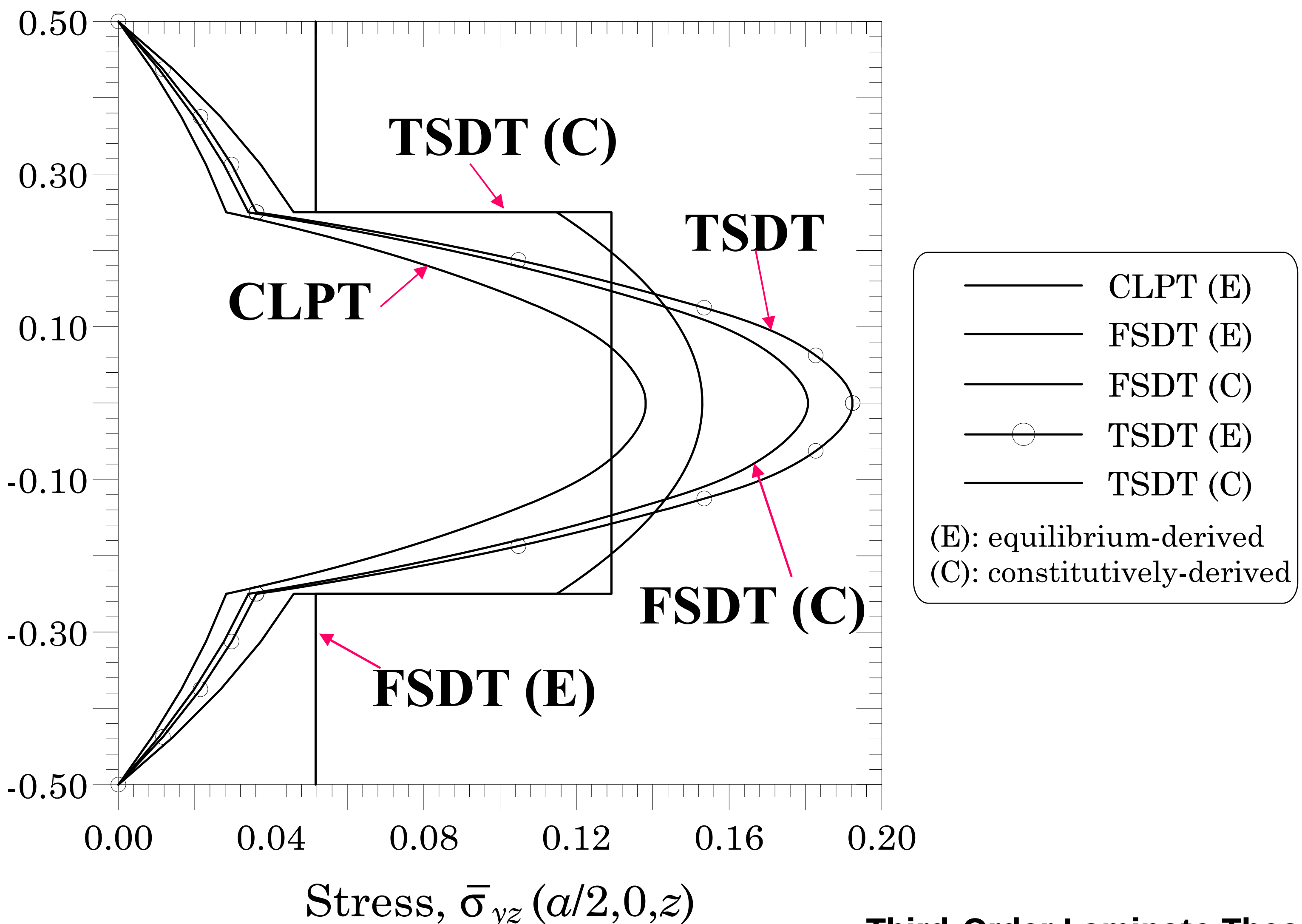
$$v(x,y,z) = v_0(x,y) + z\phi_y(x,y) - \frac{4z^3}{3h^2}\left(\phi_y + \frac{\partial w}{\partial y}\right)$$

$$w(x,y,z) = w_0(x,y)$$

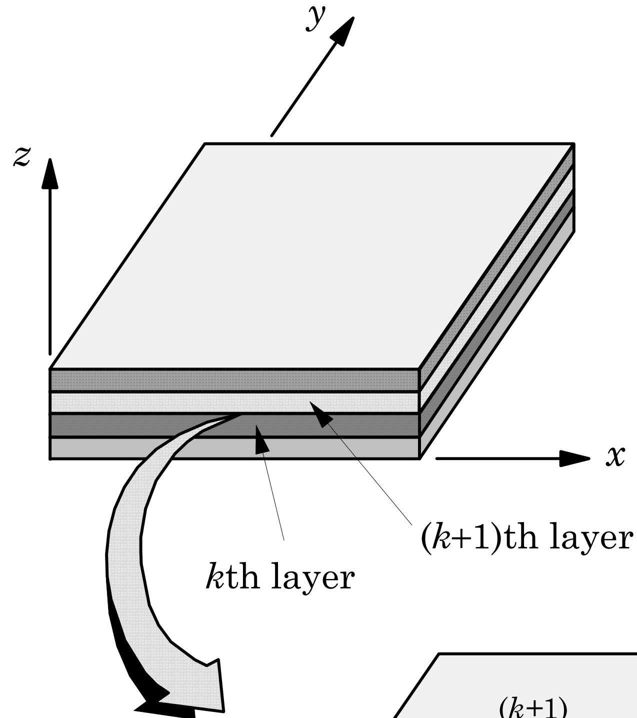
Bending of a symmetric cross-ply $(0/90)_s$ laminate (SS-1)under uniformly distributed load



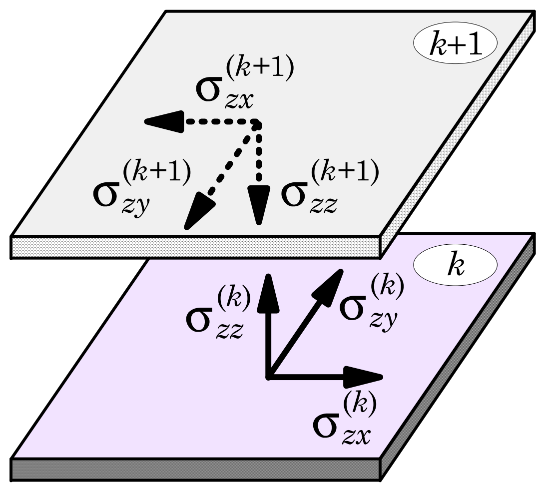
Bending of a symmetric cross-ply $(0/90)_s$ laminate (SS-1)under uniformly distributed load



LAYERWISE LAMINATE PLATE THEORY



$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix}^{k} \neq \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix}^{k+1}$$



Equilibrium of interlaminar stresses

$$\sigma_{zx}^{(k+1)} = \sigma_{zx}^{(k)}$$

$$\sigma_{zy}^{(k+1)} = \sigma_{zy}^{(k)}$$

$$\sigma_{zz}^{(k+1)} = \sigma_{zz}^{(k)}$$

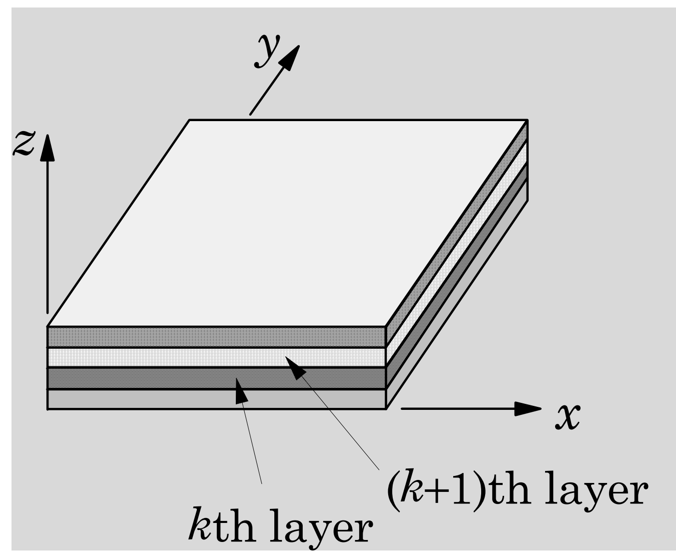
LAYERWISE LAMINATE PLATE THEORY

Equilibrium Requirements

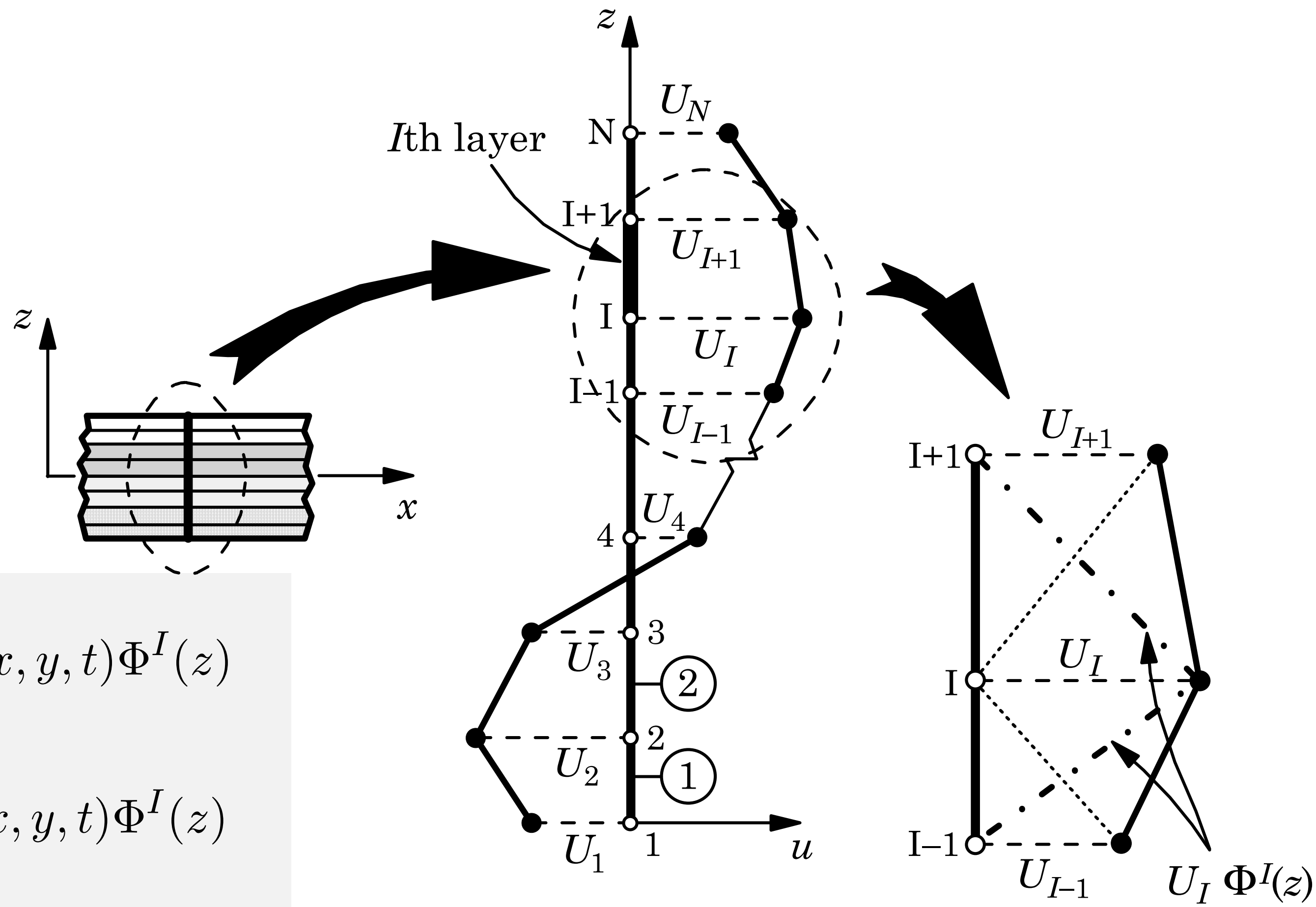
$$\begin{Bmatrix} \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \end{Bmatrix}^{k} = \begin{Bmatrix} \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \end{Bmatrix}^{k+1} \Rightarrow \begin{Bmatrix} \varepsilon_{zz} \\ \varepsilon_{yz} \\ \varepsilon_{xz} \end{Bmatrix}^{k} \neq \begin{Bmatrix} \varepsilon_{zz} \\ \varepsilon_{yz} \\ \varepsilon_{xz} \end{Bmatrix}^{k+1} \quad \text{because } Q_{ij}^{(k)} \neq Q_{ij}^{(k+1)}$$

Single-Layer Theories

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{Bmatrix}^{k} = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{Bmatrix}^{k+1}, \quad \begin{Bmatrix} \varepsilon_{zz} \\ \varepsilon_{yz} \\ \varepsilon_{xz} \end{Bmatrix}^{k} = \begin{Bmatrix} \varepsilon_{zz} \\ \varepsilon_{yz} \\ \varepsilon_{xz} \end{Bmatrix}^{k+1}$$



LAYERWISE THEORY KINEMATICS



$$u(x,y,z,t) = \sum_{I=1}^{N} U_I(x,y,t)\Phi^I(z)$$

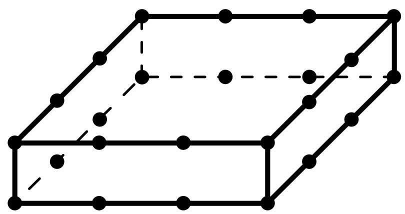
$$v(x,y,z,t) = \sum_{I=1}^{N} V_I(x,y,t)\Phi^I(z)$$

$$w(x,y,z,t) = \sum_{I=1}^{M} W_I(x,y,t)\Psi^I(z)$$

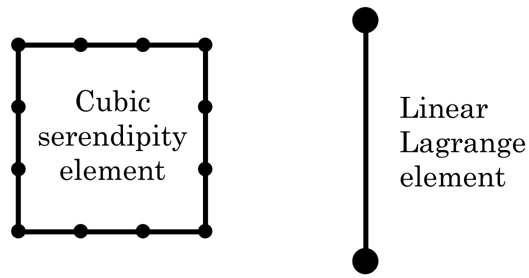
LAYERWISE KINEMATIC MODEL

Conventional 3D

Layerwise 2D + 1D

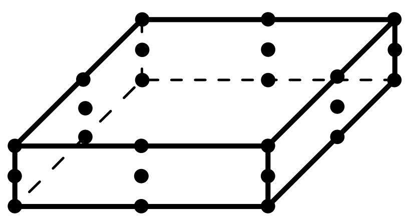


(1a)

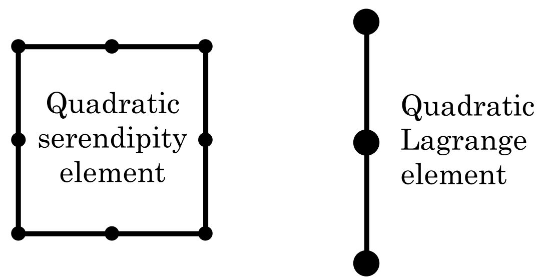


(in-plane) (through thickness)

(1b)



(2a)



(in-plane) (through thickness)

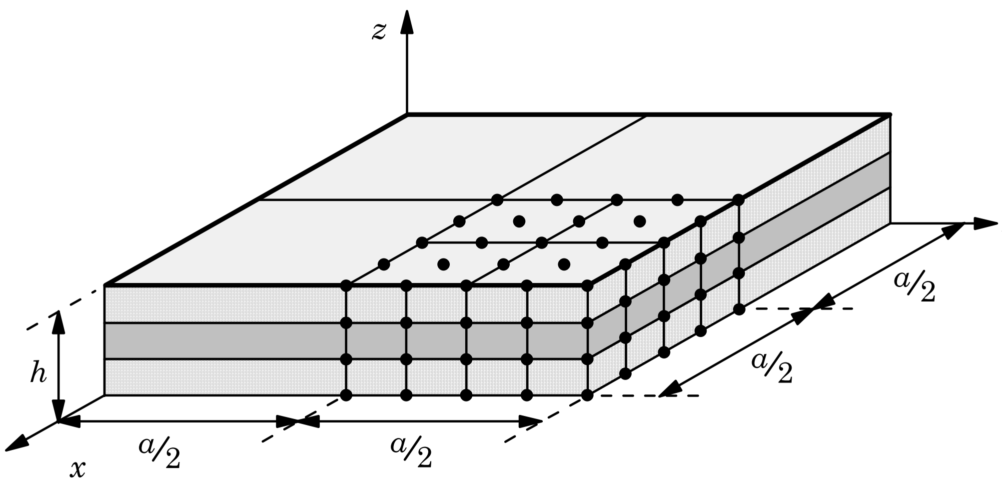
(2b)

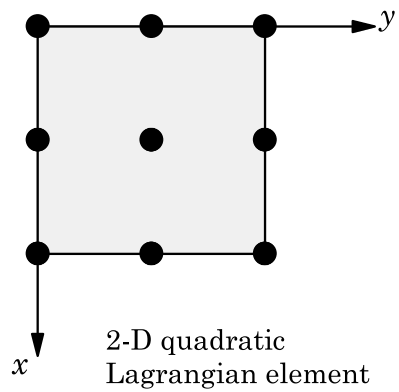
Table: Comparison of the number of operations needed to form the element stiffness matrices for equivalent elements in the conventional 3-D format and the layerwise 2-D format. Full quadrature is used in all.

Element Type†	Multipli.	Addition	Assignments
1a (3-D)	1,116,000	677,000	511,000
1b (LWPT)	423,000	370,000	106,000
2a (3-D)	1,182,000	819,000	374,000
2b (LWPT)	284,000	270,000	69,000

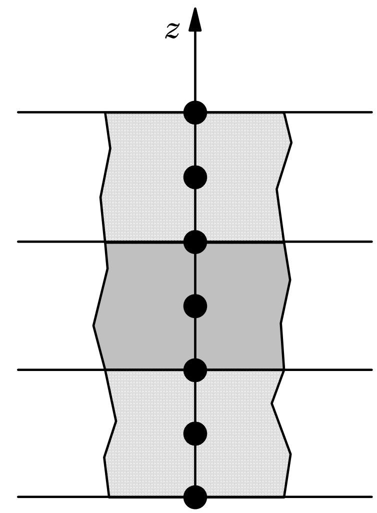
† *Element* 1*a*: 72 degrees of freedom, 24-node 3-D isoparametric hexahedron with cubic in-plane interpolation and linear transverse interpolation.
Element 1*b*: 72 degrees of freedom, E12–L1 layerwise element.
Element 2*a*: 81 degrees of freedom, 27-node 3-D isoparametric hexahedron with quadratic interpolation in all three directions.
Element 2*b*: 81 degrees of freedom, E9–Q1 layerwise element.

Layerwise Kinematic Model
3D modeling with 2D & 1D elements





2-D quadratic Lagrangian element



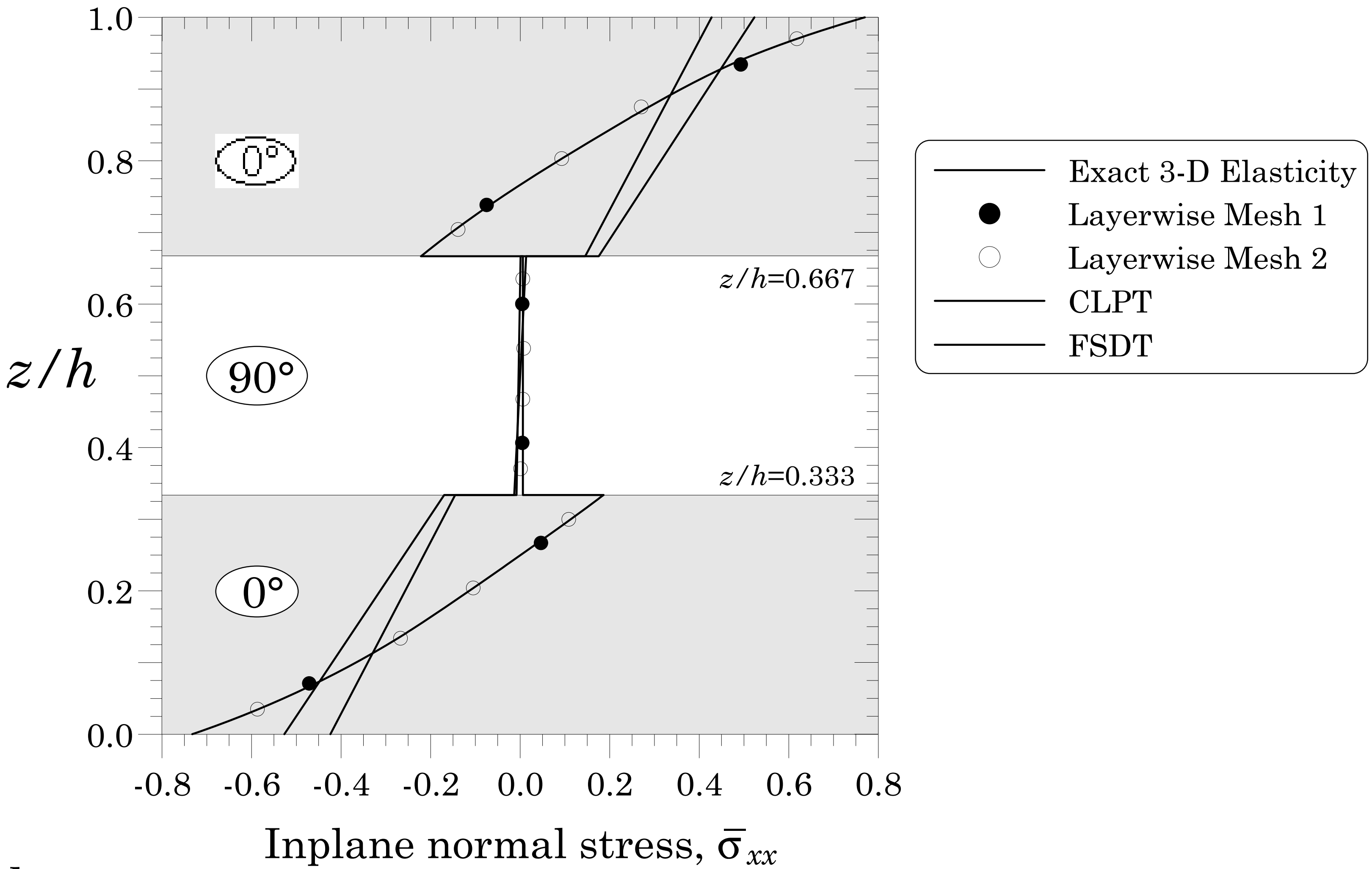
three quadratic layers through the thickness

$$E_1 = 25 \times 10^6 \text{ psi}, \quad E_2 = E_3 = 10^6 \text{ psi}$$

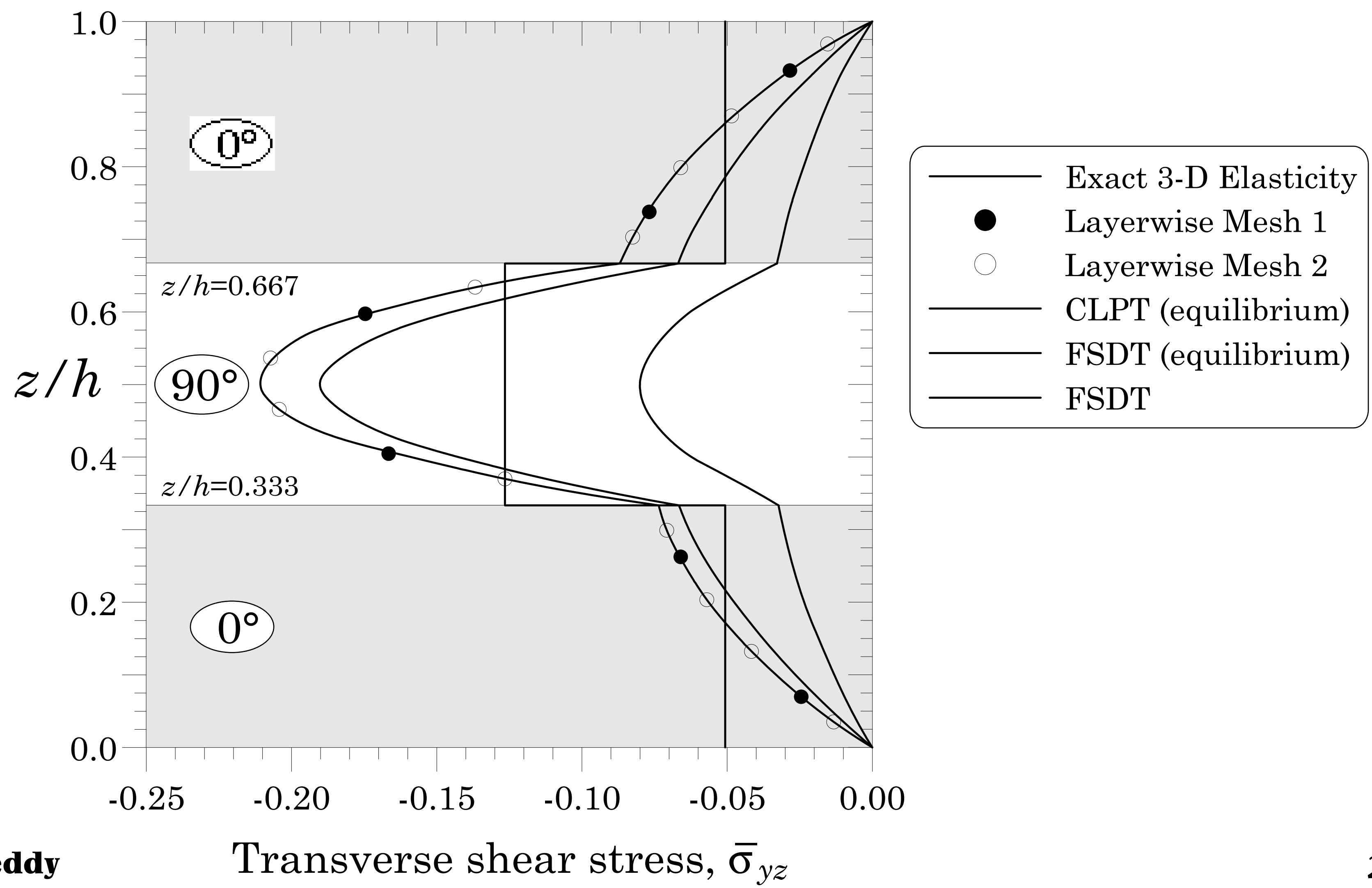
$$G_{12} = 0.5{\times}10^6 \text{ psi}, \; G_{13} = G_{23} = 0.2{\times}10^6 \text{ psi}, \; \nu_{12} = \nu_{13} = \nu_{23} = 0.25$$

$$\begin{aligned} u(x, a/2, z) &= u(a/2, y, z) = 0 \\ v(a/2, y, z) &= u(x, a/2, z) = 0 \\ w(x, a, z) &= u(a, y, z) = 0 \end{aligned}$$

In-plane Stresses predicted by the Layerwise Theory



Transverse shear stresses predicted by the Layerwise Theory



j1

Variable Kinematic Model for Global-Local Analysis

Composite displacement field:

$$u_i(x,y,z) = u_i^{ESL}(x,y,z) + u_i^{LWT}(x,y,z)$$

ESL Displacement field:

$$u_1^{ESL}(x,y,z) = u_0(x,y) + z\phi_x(x,y)$$
$$u_2^{ESL}(x,y,z) = v_0(x,y) + z\phi_y(x,y)$$
$$u_3^{ESL}(x,y,z) = w_0(x,y)$$

LWT Displacement field:

$$u_1^{LWT}(x,y,z) = \sum_{I=1}^{N} U_I(x,y)\Phi^I(z)$$

$$u_2^{LWT}(x,y,z) = \sum_{I=1}^{N} V_I(x,y)\Phi^I(z)$$

$$u_3^{LWT}(x,y,z) = \sum_{I=1}^{M} W_I(x,y)\Psi^I(z)$$

Slide 17

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j1 jnreddy, 6/30/2014

An Efficient Shell Finite Element

Objective: Develop a robust shell element for the linear and nonlinear analysis of shell structures made of multilayered composites and functionally graded materials that is computationally efficient (i.e., accurate and computationally inexpensive).

- **7-parameter displacement field**

Thickness stretch is included

$$\mathbf{u}\left(\mathbf{X}\left(\xi,\eta,\zeta\right)\right)=\bar{\mathbf{u}}\left(\xi,\eta\right)+\zeta\frac{h}{2}\boldsymbol{\varphi}\left(\xi,\eta\right)+\frac{h}{2}\zeta^{2}\Psi\left(\xi,\eta\right)\hat{\mathbf{n}}$$

- **F.E. approximation of displacement field**

$$\mathbf{u}=\sum_{k=1}^{n}\psi_{k}\left(\xi,\eta\right)\left[\bar{\mathbf{u}}^{k}+\zeta\frac{h_{k}}{2}\boldsymbol{\varphi}^{k}+\frac{h_{k}}{2}\zeta^{2}\Psi^{k}\hat{\mathbf{n}}^{k}\right]$$

NOTABLE FEATURES

- **Notable features of the 7-parameter formulation**
  - **Thickness stretching is considered**
  - **Three-dimensional constitutive equations are used**
  - **Consistent displacement finite element formulation**
- **Notable features of present implementation**
  - **Utilization of spectral/*hp* finite element technology to represent the differential geometry and avoid locking**
  - **Static condensation of degrees of freedom internal to the element**
  - **Applicability to geometrically nonlinear analysis of FGM and laminated structures**

Spectral/*hp* Finite Element Technology

Improving Numerical Efficiency: Static Condensation

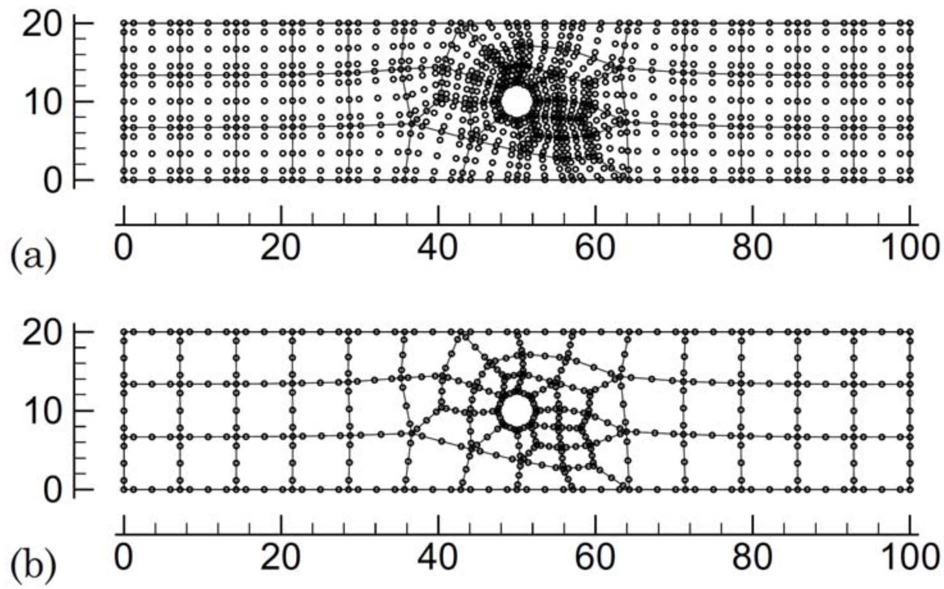


Figure: A high-order spectral/*hp* finite element discretization (*p*-level of 4) of a 2-D region: (a) finite element mesh showing elements and nodes and (b) a statically condensed version of the same mesh showing the elements and nodes.

System memory requirements for low-order and high-order problems are similar.

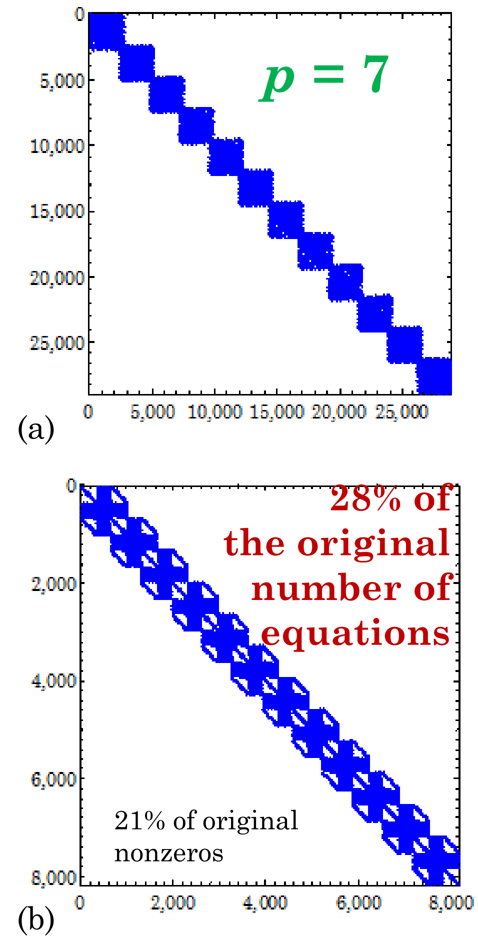
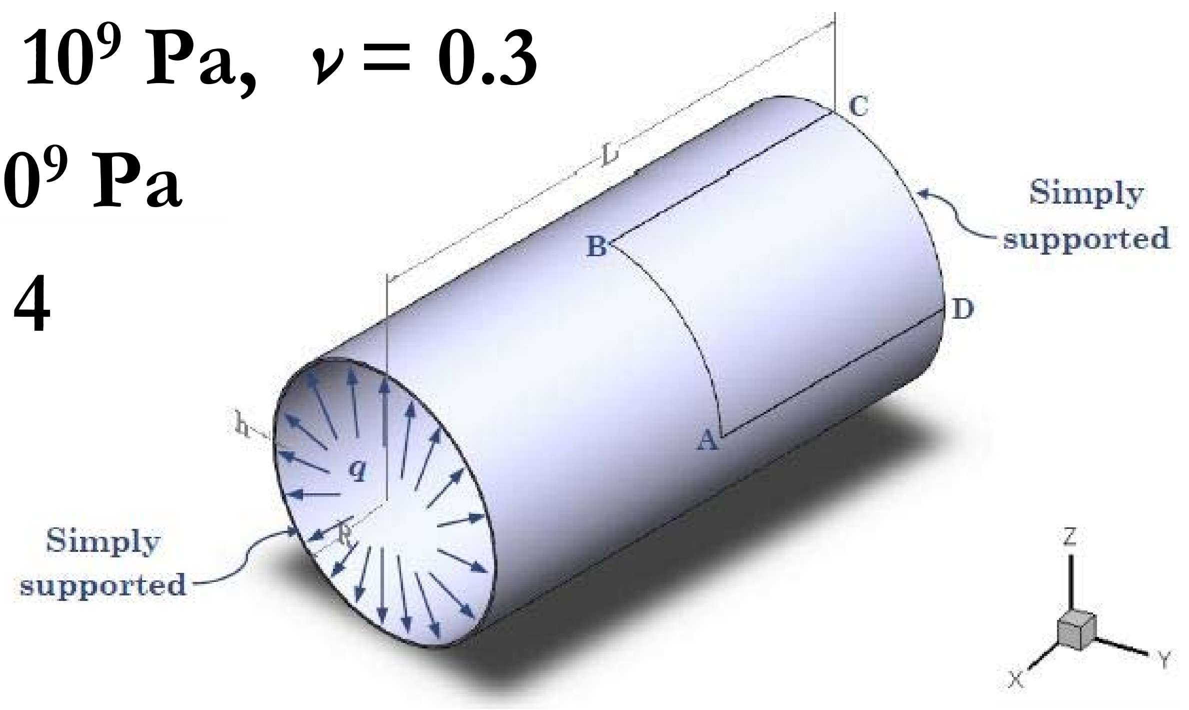


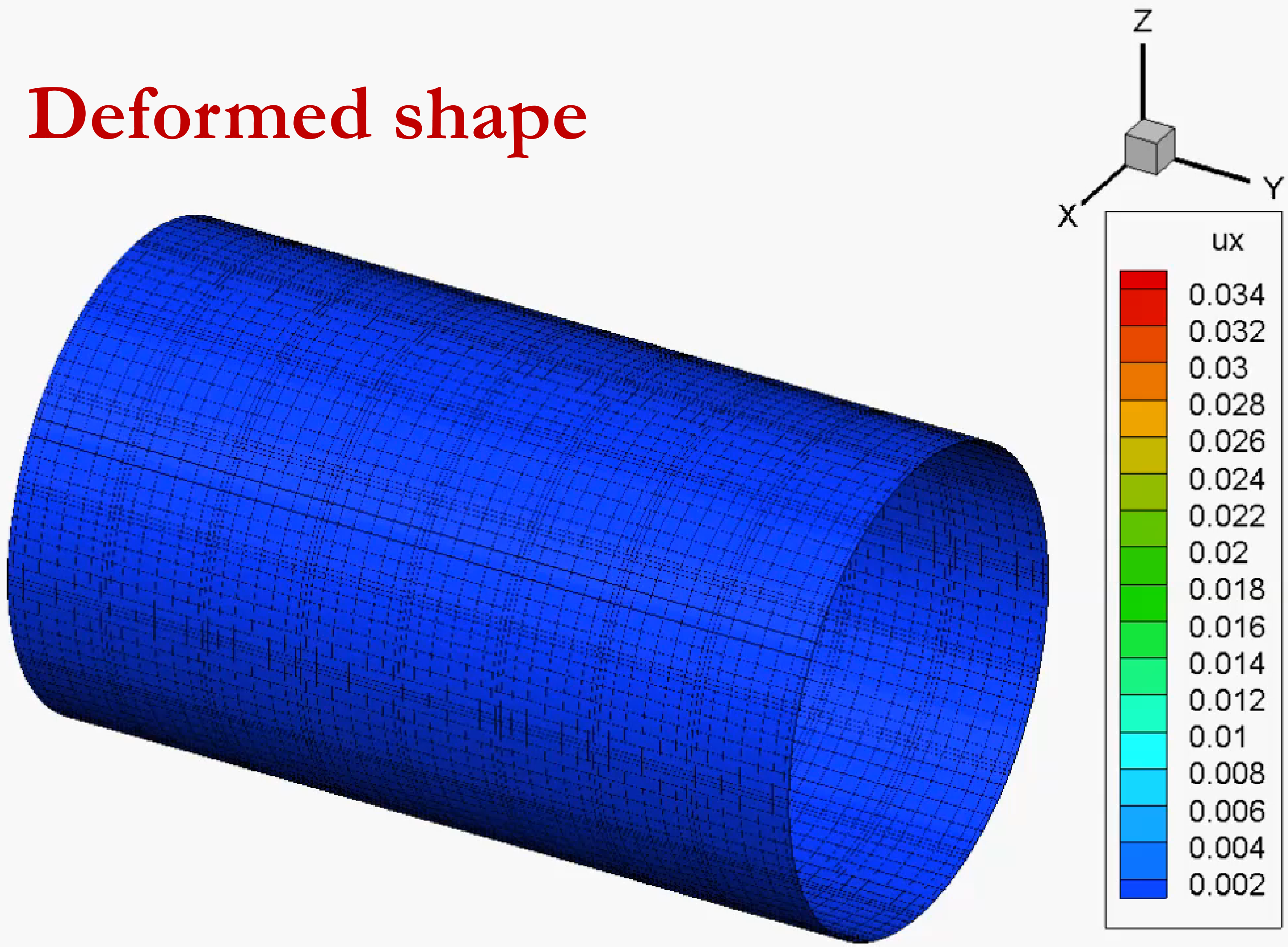
Figure: Sparsity patterns for: (a) a high-order finite element mesh and (b) the same high-order mesh using static condensation.

Benchmark Problem 1: Isotropic cylindrical shell subjected to internal pressure

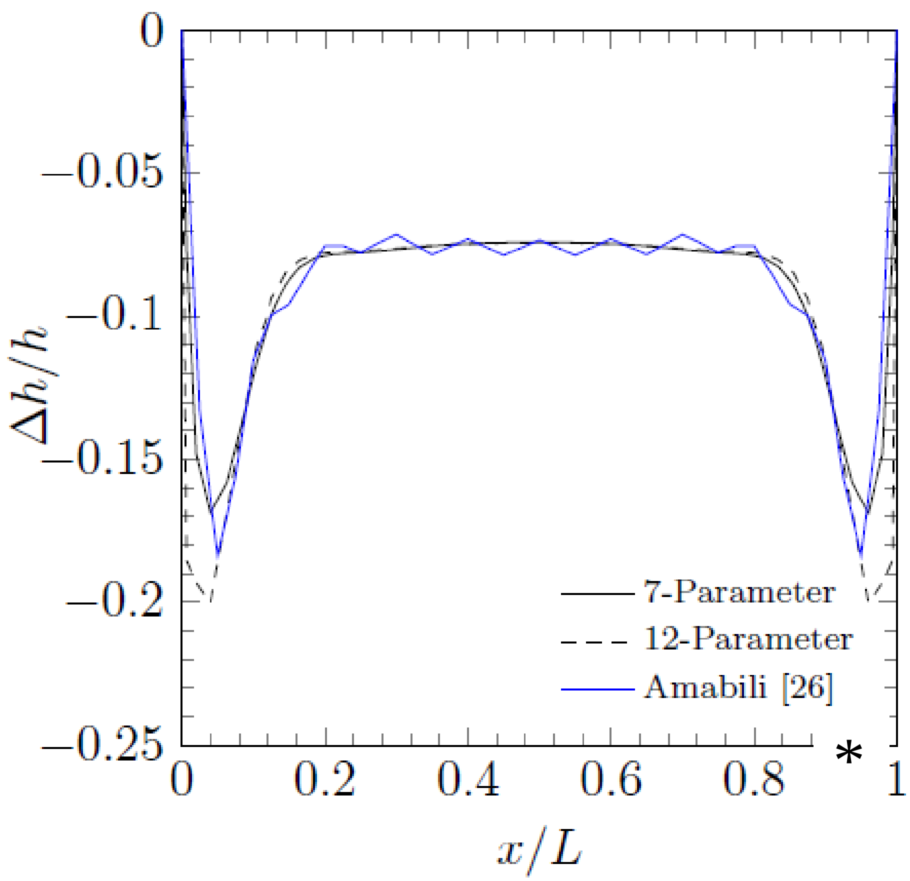
- $L = 0.52$ m, $R = 0.15$ m, $h = 0.03$ m
- $E = 198 \times 10^9$ Pa, $\nu = 0.3$
- $q = 12 \times 10^9$ Pa
- 8 x 1, $p = 4$



Deformed shape

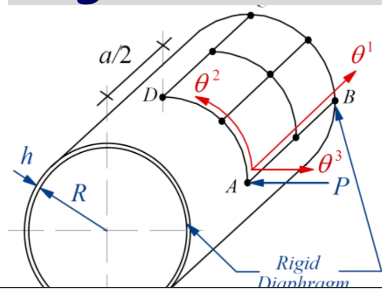


Thickness deformation vs. axial coordinate

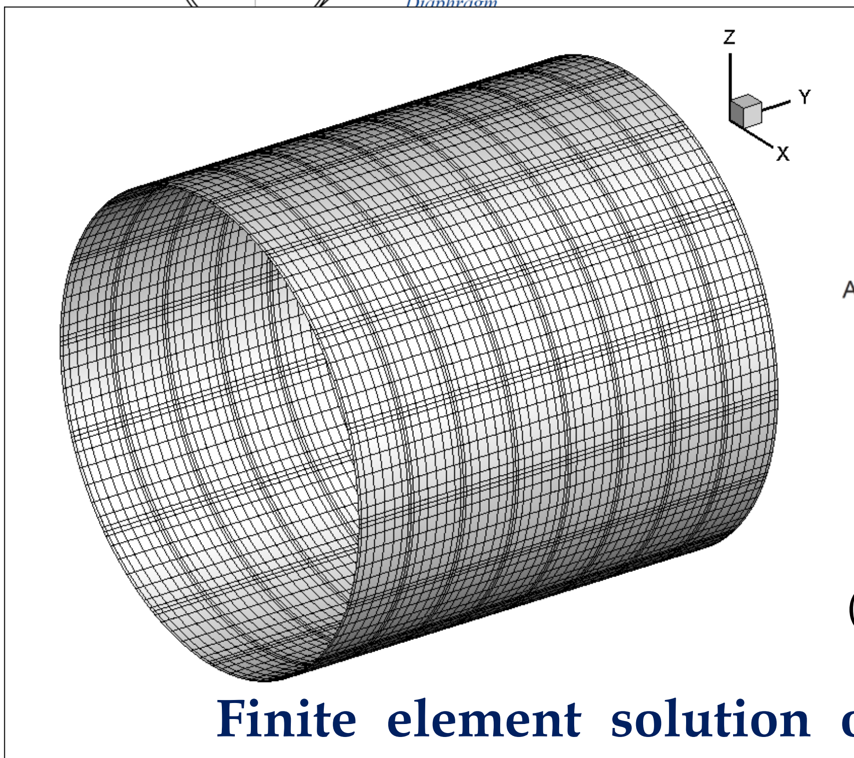


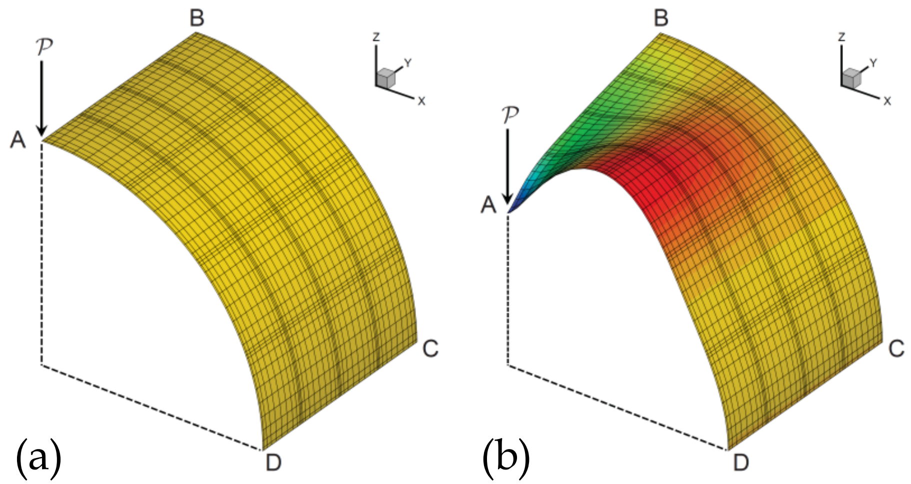
* M. Amabili, "Non-linearities in rotation and thickness deformation in a new third-order thickness deformation theory for static and dynamic analysis of isotropic and laminated doubly curved shells," International

Benchmark Problem 2: Pinched cylindrical shell



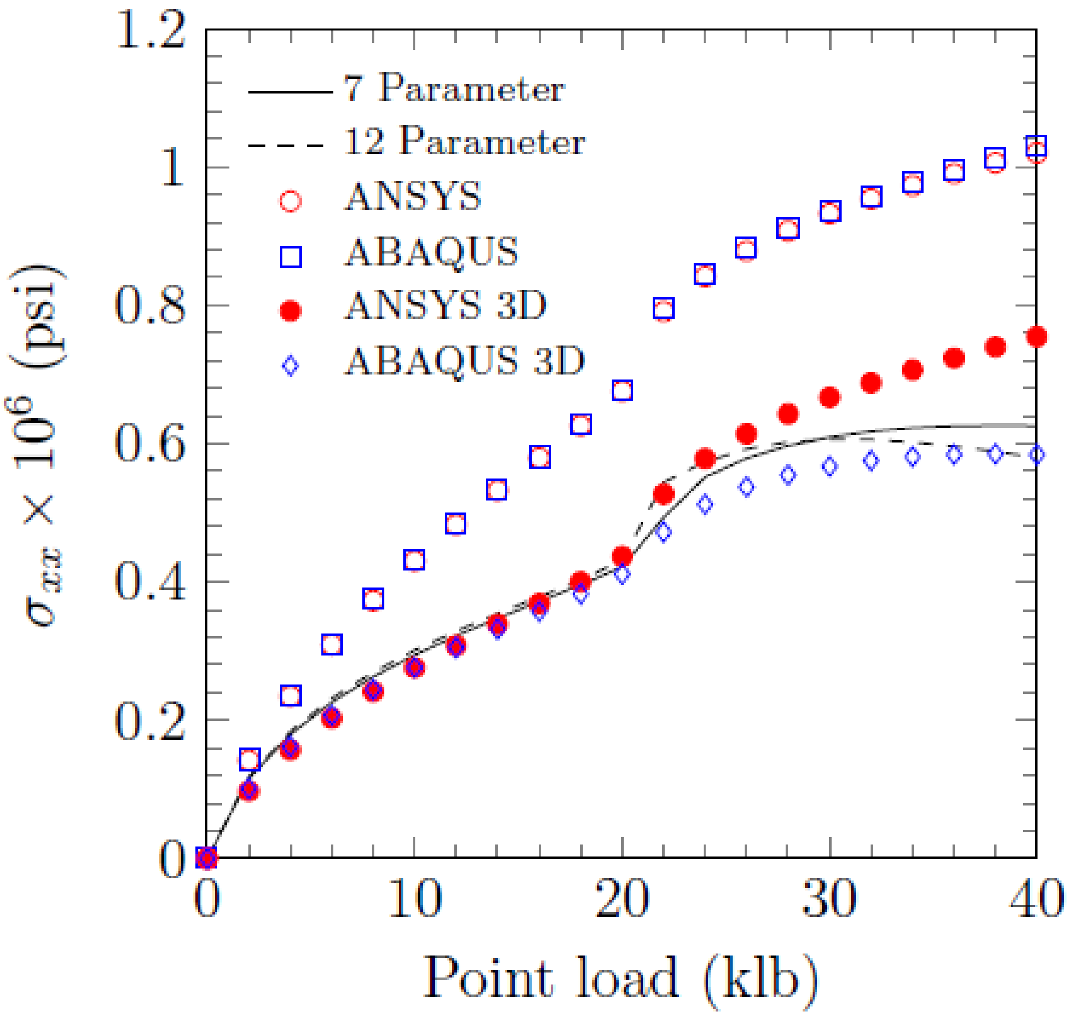
$$E = 3\times 10^6 \text{ psi}, \qquad \nu = 0.3$$
$$a = 2R = 600 \text{ in}, \qquad h = 3 \text{ in}$$
$$P = 4\mathcal{P} = 1.0 \text{ lb}_f$$





Finite element solution of deformed mid-surface of pinched cylinder. Deformation magnified by a factor of 5×10^6 (a) un-deformed shell configuration (b) deformed shell configuration.

Point load P vs. stress, σ_{xx}, at point A



Computational time

	Elements	Nodes	Degrees of freedom	Time (s)
7-parameter	4	289	2023	66
12-parameter	4	289	3468	473
ANSYS solid	13824	16807	50421	6488
ABAQUS solid	13824	16807	50421	720

THE LEAST-SQUARES METHOD

Given an operator equation of the form

$$A(u) = f \ \textbf{in} \ \Omega \quad \textbf{and} \quad B(u) = g \ \textbf{in} \ \Gamma$$

we seek suitable approximation of u as u_h. In the least-squares method, we seek the minimum of the sum of squares of the residuals in the approximation of the equations:

$$0 = \delta I(u_h) = \delta \left\{ \int_{\Omega} \left[A(u_h) - f\right]^2 d\mathbf{x} + \oint_{\Gamma} \left[B(u_h) - g\right]^2 ds \right\}$$

Variational Problem
(based on the least-squares formulation)

$$0 = \delta I(u) = \delta \left\{ \int_{\Omega} [A(u_h) - f]^2 d\mathbf{x} + \oint_{\Gamma} [B(u_h) - g]^2 ds \right\}$$

Thus, the variational problem is to seek $u_h \in H$ **such that**

$$B(\delta u_h, u_h) = \ell(\delta u_h) \text{ holds for all } \delta u_h \in H$$

where

$$B(\delta u_h, u_h) = \int_{\Omega} \delta[A(u_h)]\, A(u_h)\, d\mathbf{x} + \oint_{\Gamma} \delta[B(u_h)] B(u_h)\, ds$$

$$\ell(\delta u_h) = \int_{\Omega} \delta[A(u_h)]\, f\, d\mathbf{x} + \oint_{\Gamma} \delta[B(u_h)]\, g\, ds$$

LEAST-SQUARES FORMULATION OF VISCOUS INCOMPRESSIBLE FLUIDS

Governing equations (Navier-Stokes equations)

$$(\mathbf{u}\cdot\nabla)\mathbf{u}+\nabla p-\frac{1}{\mathrm{Re}}\nabla\cdot[(\nabla\mathbf{u})+(\nabla\mathbf{u})^T]=\mathbf{f}\quad\text{in}\quad\Omega$$

$$\nabla\cdot\mathbf{u}=0\quad\text{in}\quad\Omega$$

$$\mathbf{u}=\hat{\mathbf{u}}\quad\text{on}\quad\Gamma_{\mathrm{u}}$$

$$\hat{\mathbf{n}}\cdot\boldsymbol{\sigma}=\hat{\mathbf{t}}\quad\text{on}\quad\Gamma_{\sigma}$$

VELOCITY-PRESSURE-VORTICITY FORMULATION

$$(\mathbf{u}\cdot\nabla)\mathbf{u}+\nabla p-\frac{1}{\mathrm{Re}}\nabla\times\boldsymbol{\omega}=\mathbf{f}\quad\text{in}\quad\Omega$$

$$\boldsymbol{\omega}-\nabla\times\mathbf{u}=\mathbf{0}\quad\text{in}\quad\Omega$$

$$\nabla\cdot\mathbf{u}=0\quad\text{in}\quad\Omega$$

$$\nabla\cdot\boldsymbol{\omega}=0\quad\text{in}\quad\Omega$$

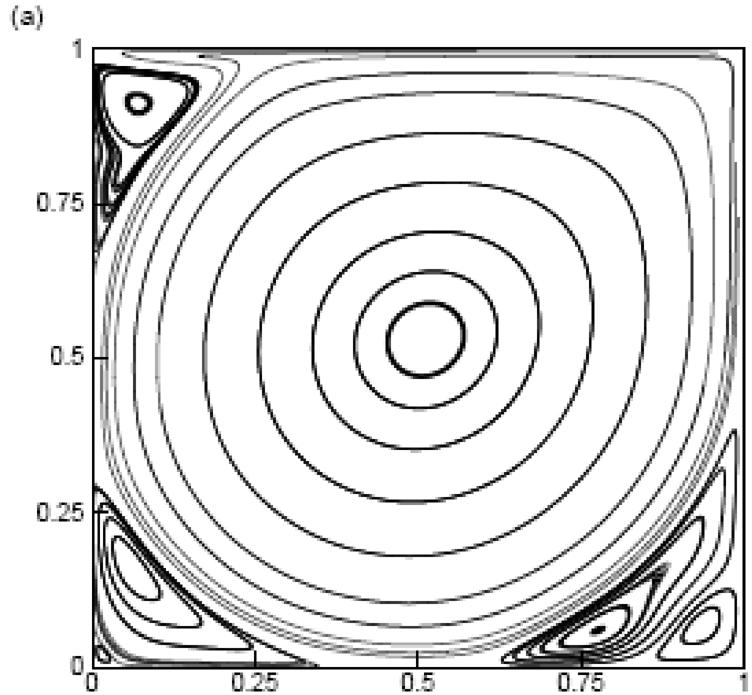
$$\mathbf{u}=\hat{\mathbf{u}}\quad\text{on}\quad\Gamma_{\mathrm{u}}$$

$$\boldsymbol{\omega}=\hat{\boldsymbol{\omega}}\quad\text{on}\quad\Gamma_{\omega}$$

$$\mathcal{J}(\mathbf{u},p,\omega;\mathbf{f})=\frac{1}{2}\Big(\|(\mathbf{u}\cdot\nabla)\mathbf{u}+\nabla p+\frac{1}{\mathrm{Re}}\nabla\times\omega-\mathbf{f}\|_0^2+\|\omega-\nabla\times\mathbf{u}\|_0^2$$
$$+\|\nabla\cdot\mathbf{u}\|_0^2+\|\nabla\cdot\omega\|_0^2\Big)$$

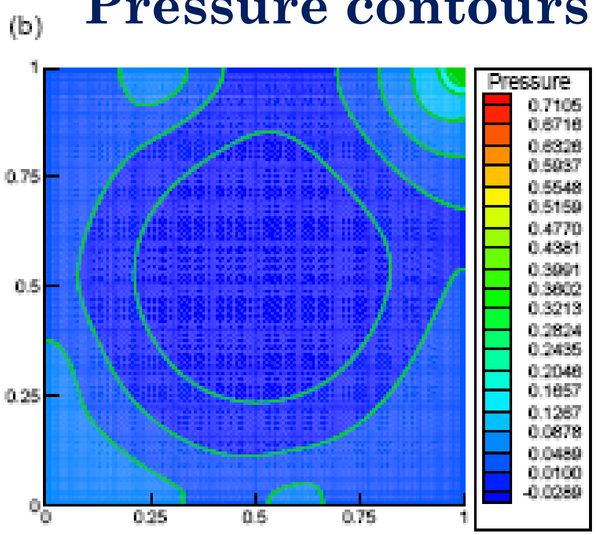
Lid-Driven Cavity Problem

$$Re = 10^4$$

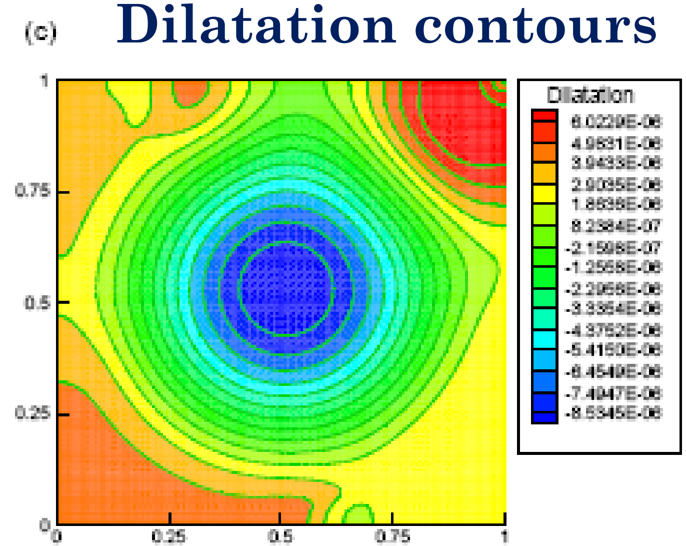


Streamlines

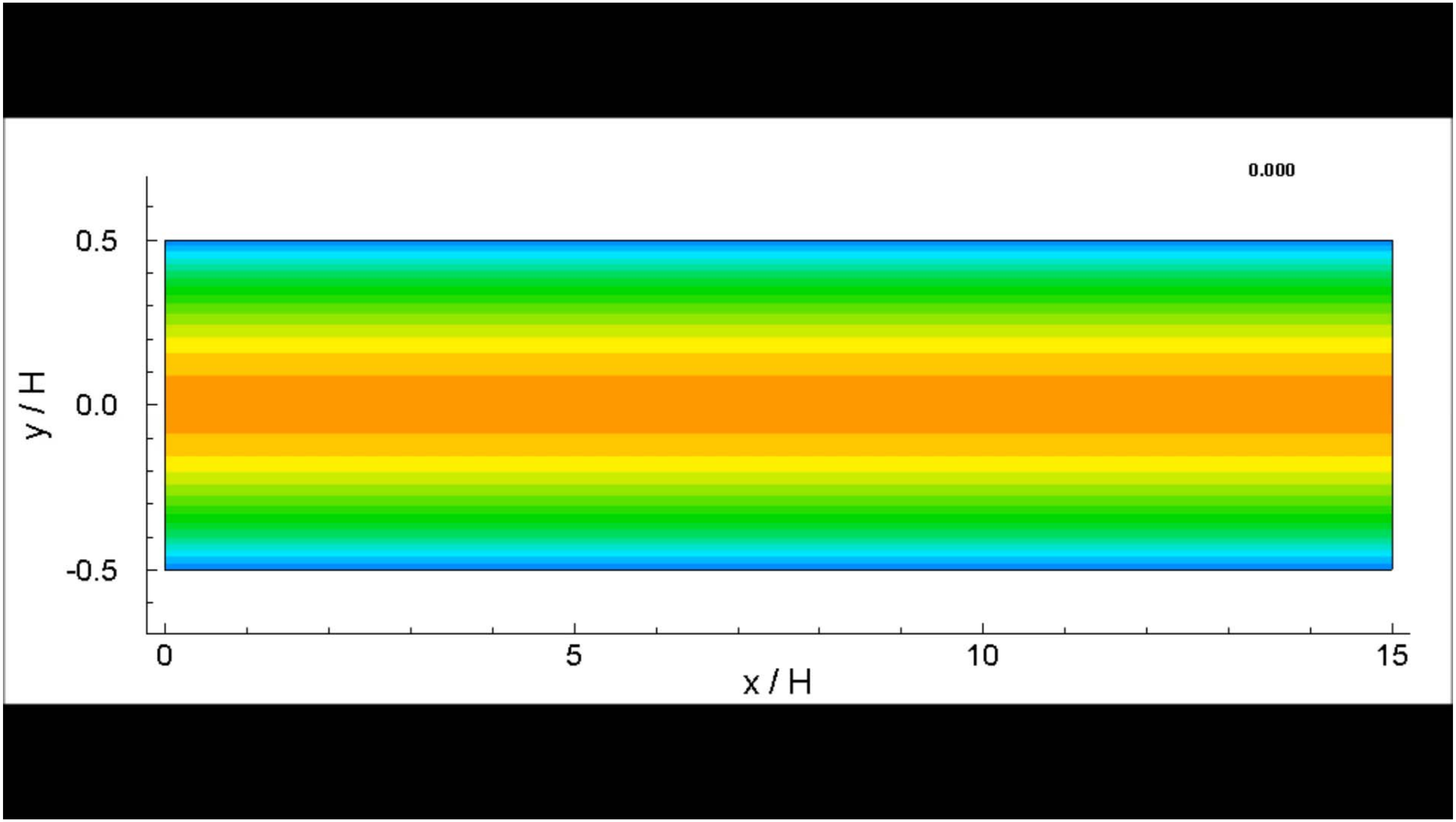
Pressure contours



Dilatation contours



RESULTS OF OTHER NON-TRIVIAL FLOW PROBLEMS



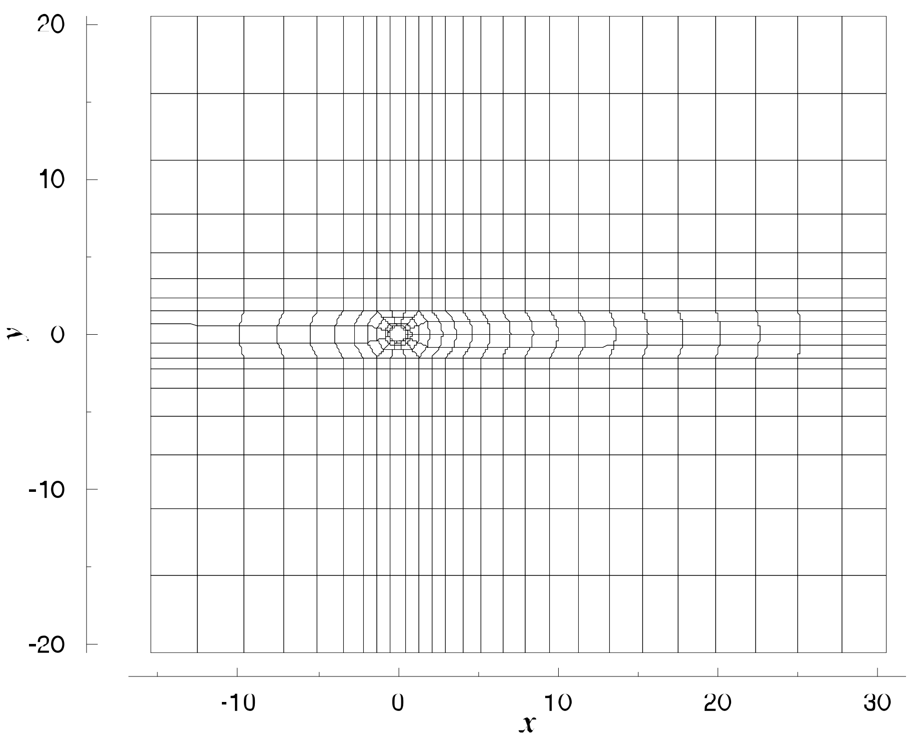
JN Reddy

Flow of a Viscous Incompressible Fluid around a Cylinder-1

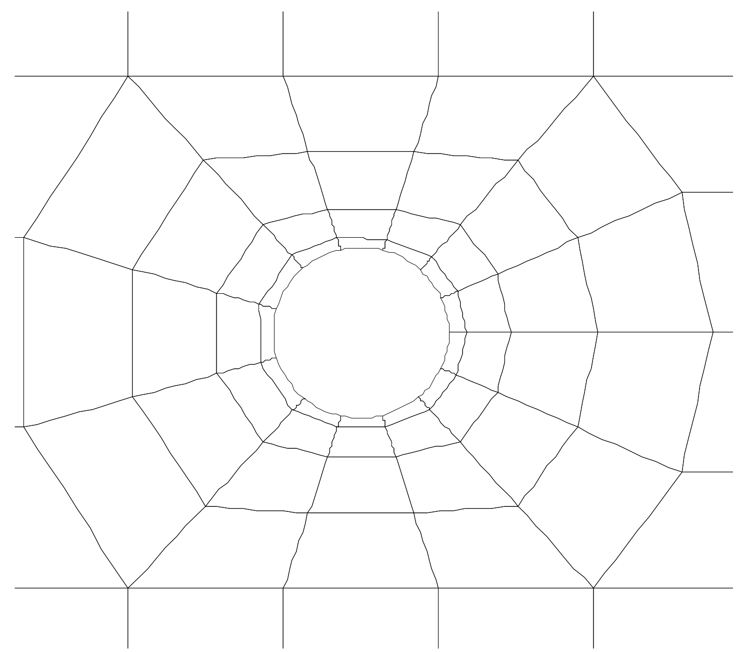
Mesh (501 elements; *p*=4)

Close-up of mesh around the cylinder

(a)

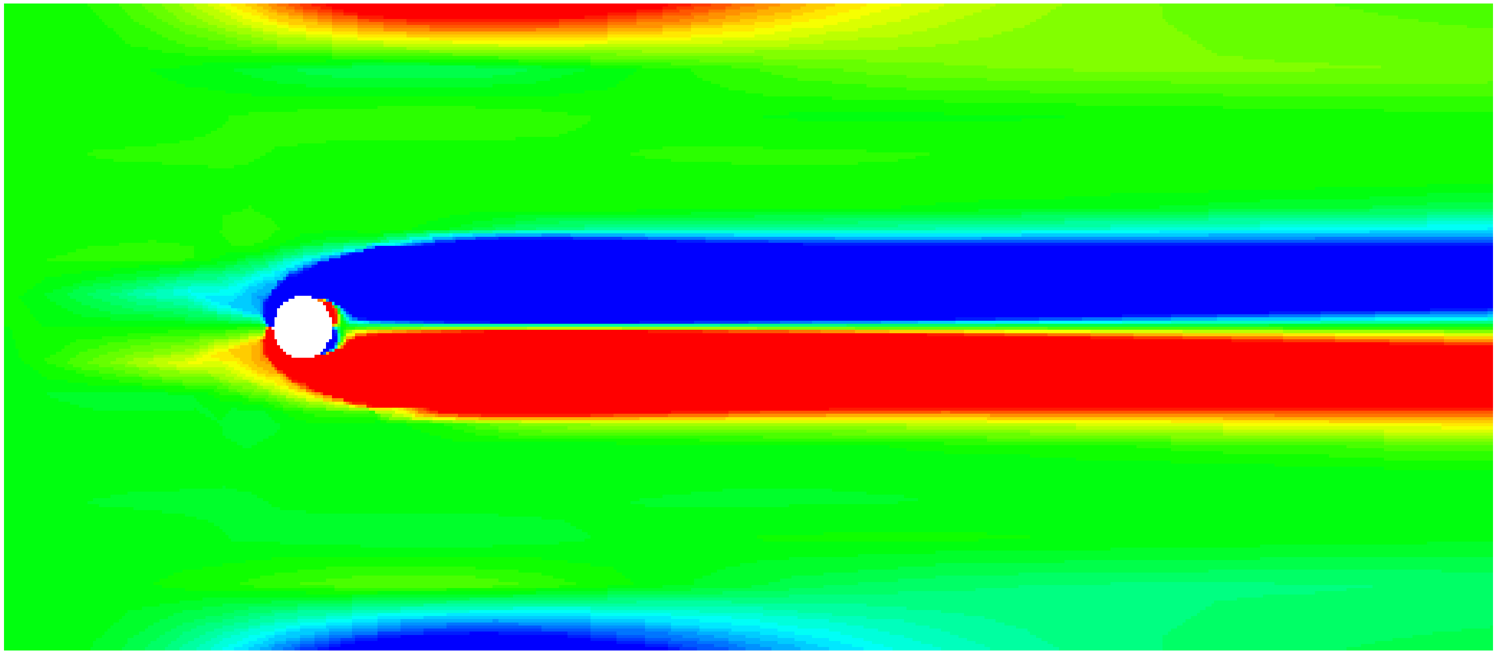


(b)



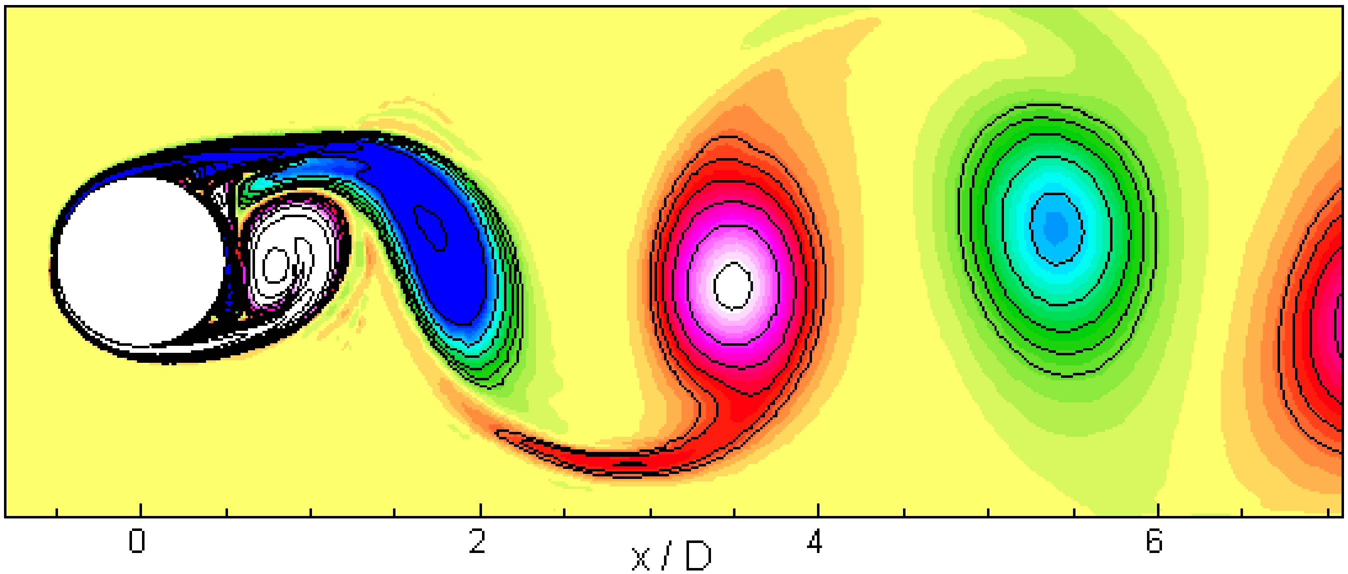
Circular Cylinder in Crossflow

Vorticity Contours



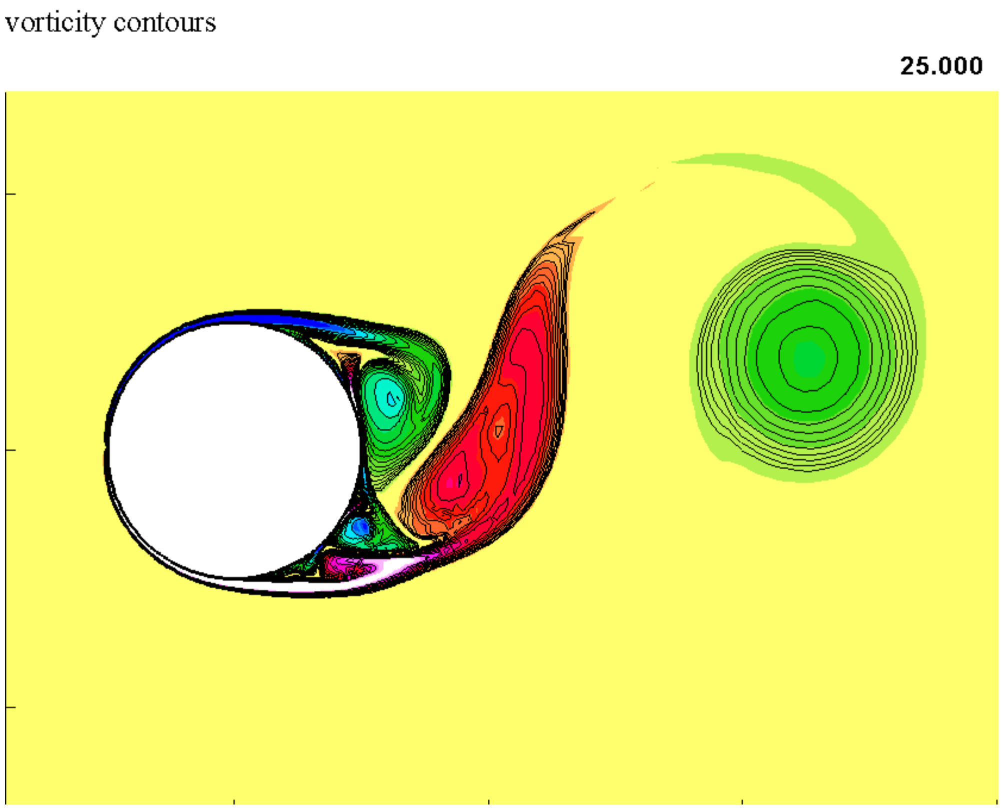
Non-stationary incompressible N-S equations, Re = 100
Least-Squares time / space decoupled formulation
1200 elements with p = 2

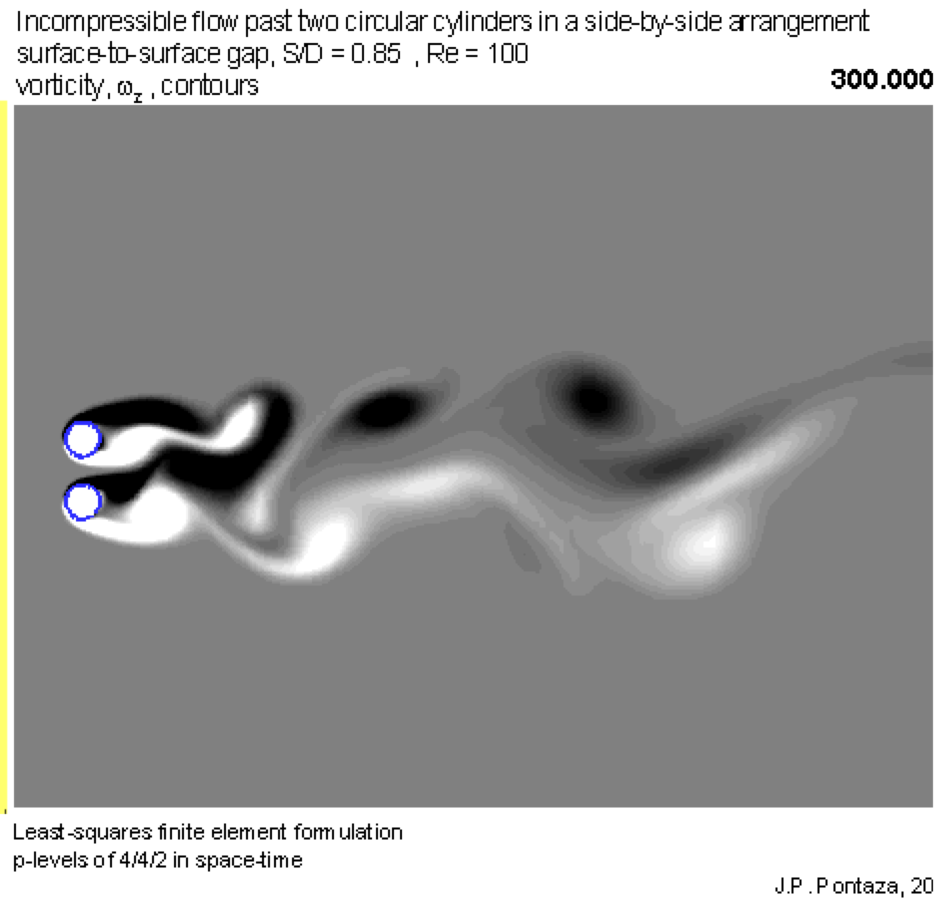
2D Flows Past a Circular Cylinder-2



- Robust at moderately high Reynolds numbers: Re = 100 – 10^4
- High *p*-level solution: $p = 4, 6, 8, 10$
- No filters or stabilization are needed

Flow of a viscous fluid past a circular cylinder-3





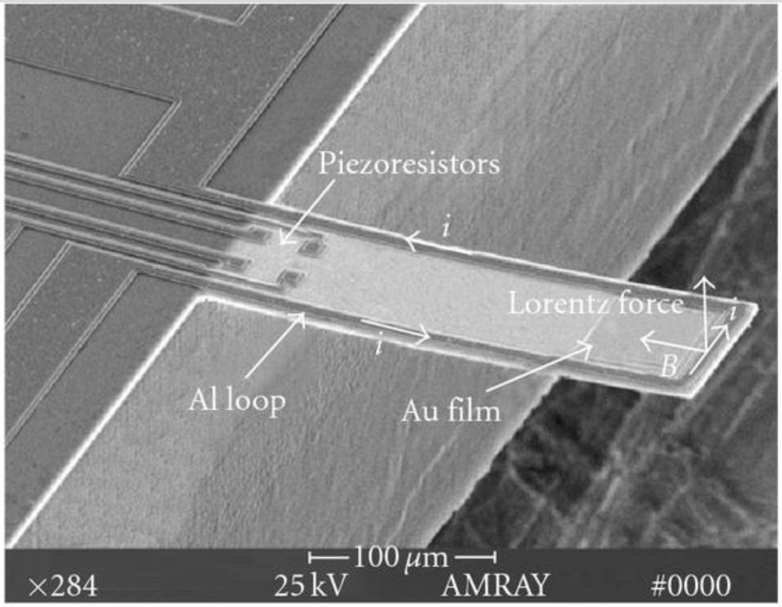
NON-LOCAL AND NON-CLASSICAL CONTINUUM MECHANICS

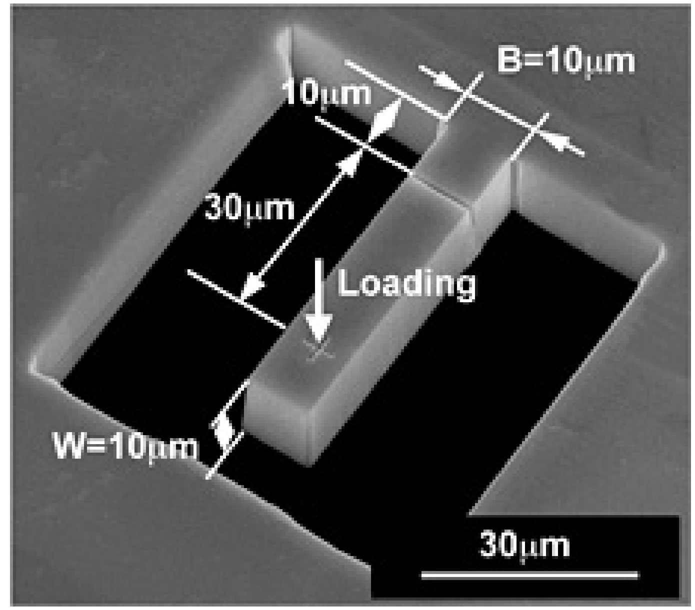
Non-locality can arise from the way we choose to model physical phenomena.

Some of the ways the non-locality is modeled are:

- **Cosserat or micropolar continuum,**
- **Strain gradient theories and Modified couple stress theories,**
- **Eringen's integral, differential, and integro-differential models, and**
- **Peridynamics, which is an integral representation of balance laws accounting for long-range forces.**

MICRO- AND NANO-ELECTRO-MECHANICAL SYSTEMS





Normalized bending stiffness increases as the cantilever beam thickness decreases. Measurable at micron-order thicknesses. (McFarland & Colton, 2005)

$$\delta = \frac{PL^3}{3EI} \Rightarrow k = \frac{P}{\delta} = \frac{3EI}{\varphi L^3} = \frac{Ewh^3}{4\varphi L^3}$$

$(\varphi = 1$, for plane stress; $\varphi = 1 - \nu^2$ for plane strain$)$

Biomechanics – Bones

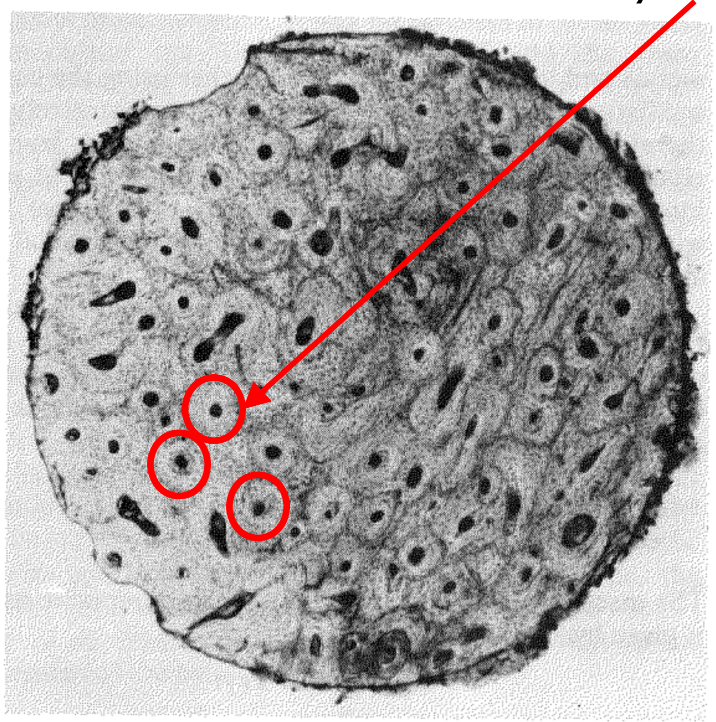
Dynamical Study of Couple Stress Effects in Human Compact Bone

. S. Lakes

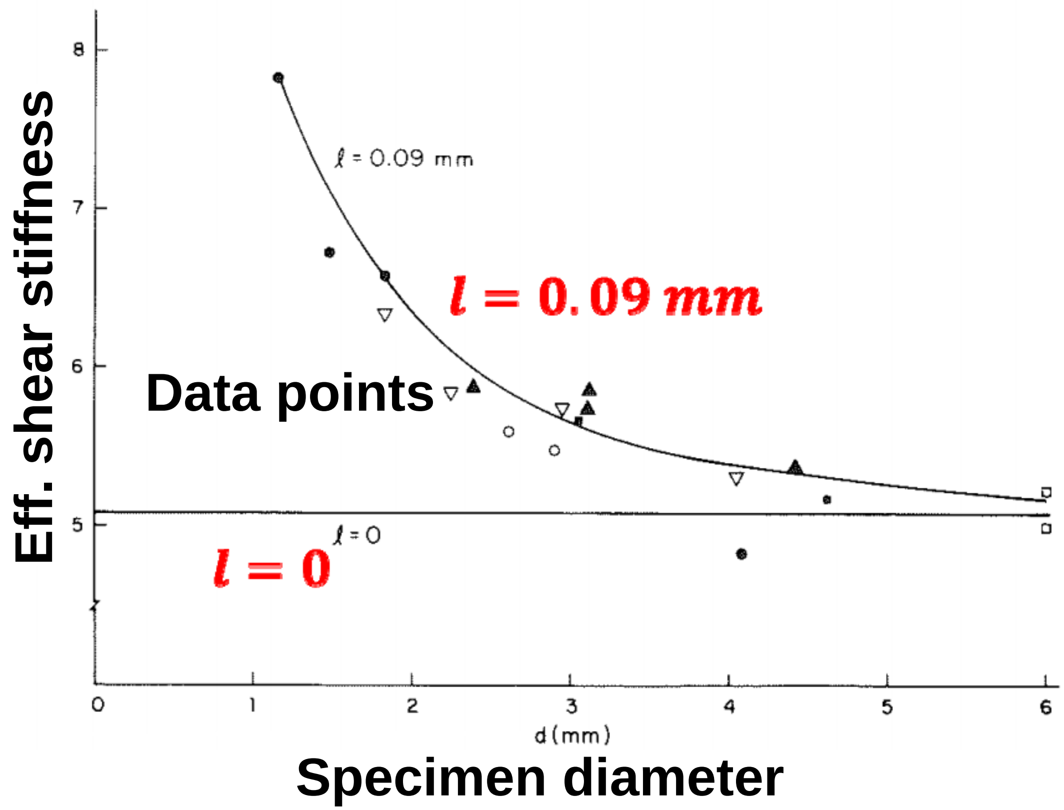
Torsional resonance experiments performed on wet human compact bone disclose effects due to couple stress. The characteristic length, which is an additional material coefficient which appears in couple-stress theory, is of the order of the size of osteons and appears to be smaller at high frequencies than at low frequencies. The presence of couple-stress effects implies a reduction in the stress concentration factor around holes, particularly small holes.

***Journal of Biomechanical Engineering* (1982)**

Osteons, *d* = 0.1 or 0.2 mm



g. 5 Transmitted light micrograph of a typical specimen. Specimen ameter is 2.62 mm.



LAKE's USE OF MICROPOLAR THEORY
TO EXPLAIN NONLOCAL EFFECTS

Classical continuum

$$\sigma_{ij} = 2\mu\,\varepsilon_{ij} + \lambda\delta_{ij}\,\varepsilon_{kk}$$

Cosserat continuum (Cosserats, 1909)

$$\sigma_{ij} = \left(2\mu + \kappa\right)\varepsilon_{ij} + \lambda\delta_{ij}\,\varepsilon_{kk} + \kappa e_{ijm}(\omega_m - \phi_m)$$

$$m_{ij} = \alpha\,\phi_{k,k}\,\delta_{ij} + \beta\phi_{i,j} + \gamma\phi_{j,i}$$

κ, α, β, and γ are micropolar constants.

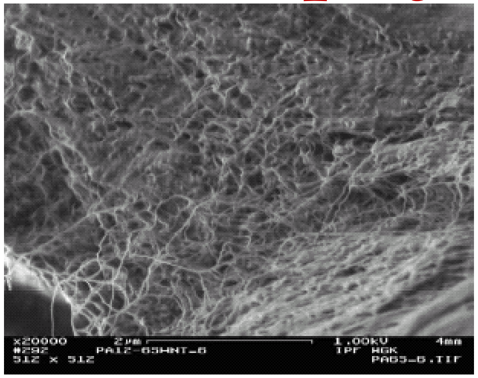
Professor Karan Surana has a presentation on some elements of non-classical continuum mechanics

STRAIN GRADIENT ELASTICITY THEORY
(Srinivasa & Reddy, JMPS, 2013)

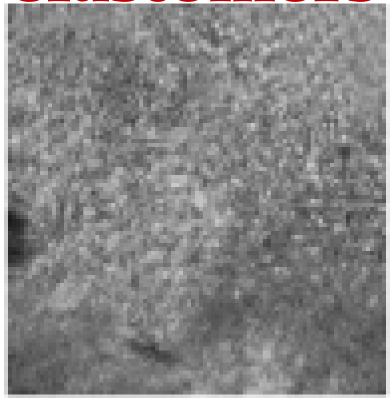
Why rotational gradient dependent elasticity?

Presence of very stiff secondary phases giving rise to distinct microscopic length scale. Intuitively, the secondary phase "rotates" with the material but does not stretch with it; interference between neighbors causes it to resist rotational gradients leading to couple stresses.

CNT reinforced polymers and nematic elastomers



PA12 with 6% SWNT (showing network formation) introducing distinct microscopic length scale



Nematic elastomer with hard nematic phase of random orientation embedded in a soft polymer matrix

GRADIENT ELASTICITY THEORY
(Srinivasa & Reddy, JMPS)

Conventional stress

$$S_{\alpha\beta} = \frac{\partial \Psi}{\partial E_{\alpha\beta}}, \qquad S_{33} = \frac{\partial \Psi}{\partial E_{33}}$$

"Bending" couple stress

$$\tau_{\alpha} = \frac{\partial \Psi}{\partial \omega_{,\alpha}}$$

"Drilling" couple stress

$$T_{\alpha\beta} = \frac{\partial \Psi}{\partial w_{,\alpha\beta}}$$

Gradient dependent terms

Governing equations in terms of stress resultants :

$$N_{\alpha\beta,\beta} - e_{\alpha\gamma 3}\,\tau_{\beta,\beta\gamma} = 0$$

$$M_{\alpha\beta,\alpha\beta} - \left[\left(N_{\alpha\beta} + N_{33}\delta_{\alpha\beta}\right) w_{,\beta}\right]_{,\alpha} = 0$$

TIMOSHENKO BEAM THEORY

$$0=\int_0^L\int_A\left(\sigma_{xx}\delta\varepsilon_{xx}+2\sigma_{xz}\varepsilon_{xz}+\sigma_{zz}\delta\varepsilon_{zz}+2m_{xy}\delta\chi_{xy}\right)dAdx$$

$$-\int_0^L\left(q\delta w+f\delta u\right)dx$$

$$-\frac{d}{dx}\left(N_{xx}\right)-f=0,\quad -\frac{dM_{xx}}{dx}+Q_x=0$$

$$-\frac{d}{dx}\left(Q_x+N_{xx}\frac{dw}{dx}+\frac{d\mathfrak{M}}{dx}\right)-q=0$$

Mindlin model

$$\mathfrak{M}=2G\ell^2\frac{d\theta_x}{dx}$$

the square root of the ratio of the moduli of curvature to the shear (a property measuring the effect of the couple stress)

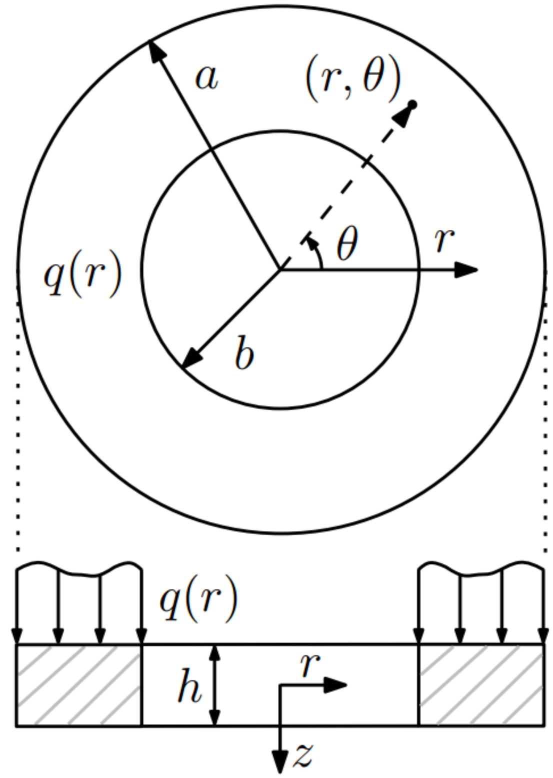
Srinivasa-Reddy model

$$\mathfrak{M}=\alpha E_{11}+2\gamma E_{13}+\lambda\frac{d\theta_x}{dx}$$

Needs to be interpreted in the context of a specific problem

Microstructure-dependent (gradient elasticity) Mindlin plate

$$U_r(r,z) = z\phi(r)\ , \quad U_z(r,z) = w(r)$$



$$\epsilon_r = \frac{\partial U_r}{\partial r} = z\frac{\partial \phi}{\partial r}\ , \quad \epsilon_\theta = \frac{U_r}{r} = \frac{z}{r}\phi\ ,$$

$$\gamma_{rz} = \frac{\partial U_r}{\partial z} + \frac{\partial U_z}{\partial r} = \phi + \frac{\partial w}{\partial r}$$

$$\chi_{r\theta} = \frac{1}{2}\left(\frac{\partial \omega_\theta}{\partial r} - \frac{\omega_\theta}{r}\right) = \frac{1}{4}\left[\frac{\partial \phi}{\partial r} - \frac{\partial^2 w}{\partial r^2} - \frac{1}{r}\left(\phi - \frac{\partial w}{\partial r}\right)\right]$$

$$\sigma_r = \frac{E(z)}{1-\nu^2}(\epsilon_r + \nu\epsilon_\theta)\ , \quad \sigma_\theta = \frac{E(z)}{1-\nu^2}(\nu\epsilon_r + \epsilon_\theta)$$

$$\tau_{rz} = G(z)\gamma_{rz}\ , \quad m_{r\theta} = 2G(z)l^2\chi_{r\theta}$$

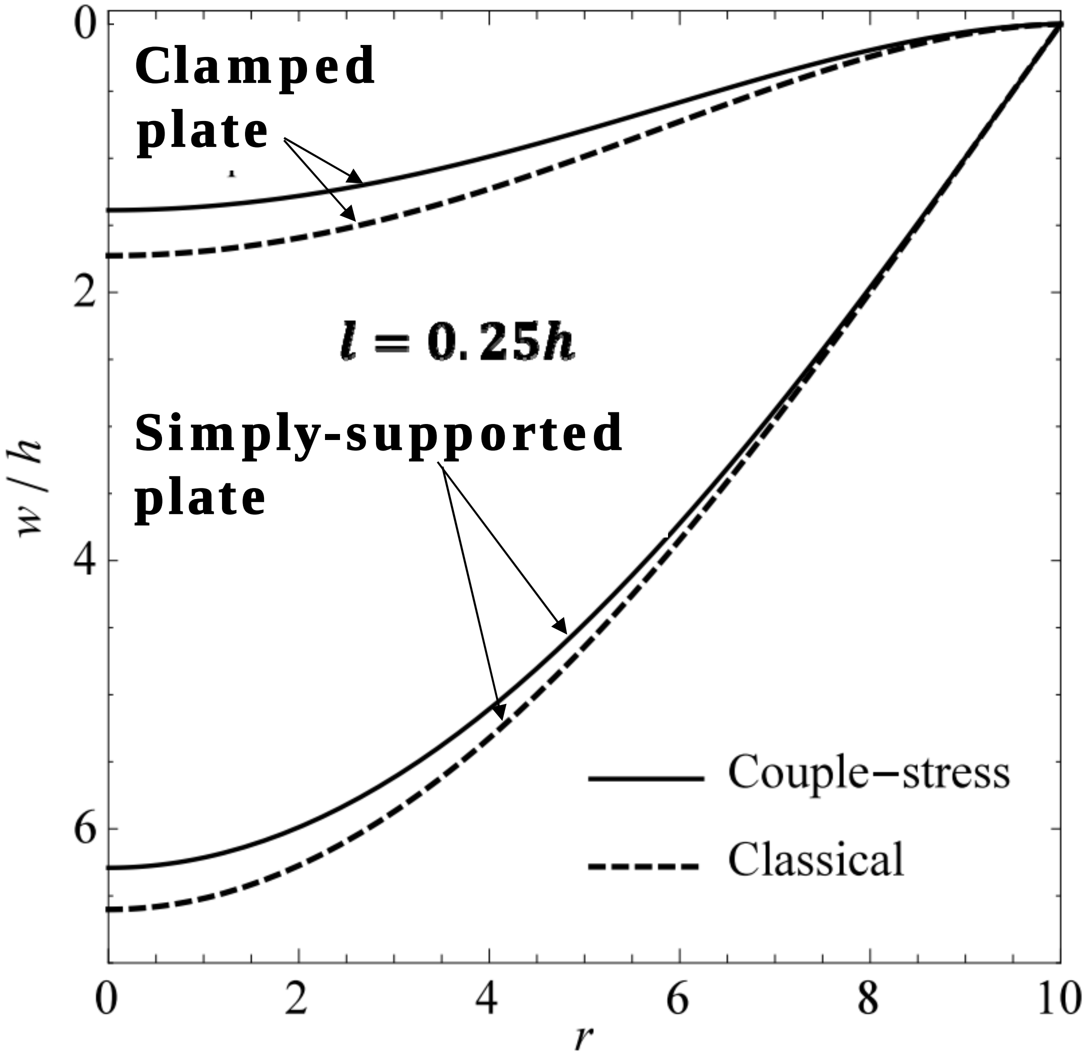
$$P_{r\theta}(r) = \int_{-h/2}^{h/2} m_{r\theta}\,dz = \frac{1}{2}S_{r\theta}\left[\frac{\partial \phi}{\partial r} - \frac{\partial^2 w}{\partial r^2} - \frac{1}{r}\left(\phi - \frac{\partial w}{\partial r}\right)\right]$$

JN Reddy

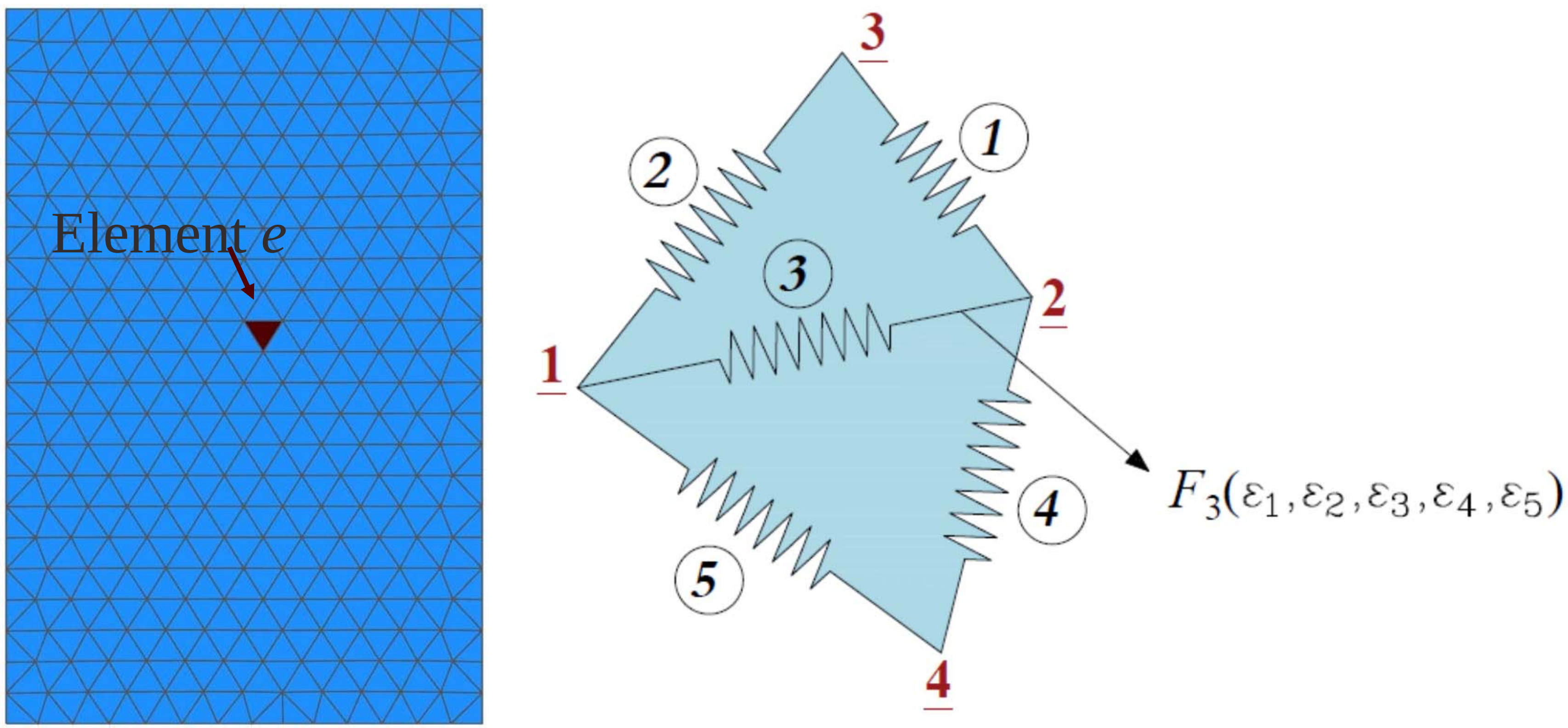
Bending of a solid circular plate

$$r = a: \; w = \phi = \theta = 0 \,, \quad \textbf{Clamped}$$

$$r = a: \; w = \hat{M} = \hat{P} = 0 \quad \textbf{Simply-supported}$$

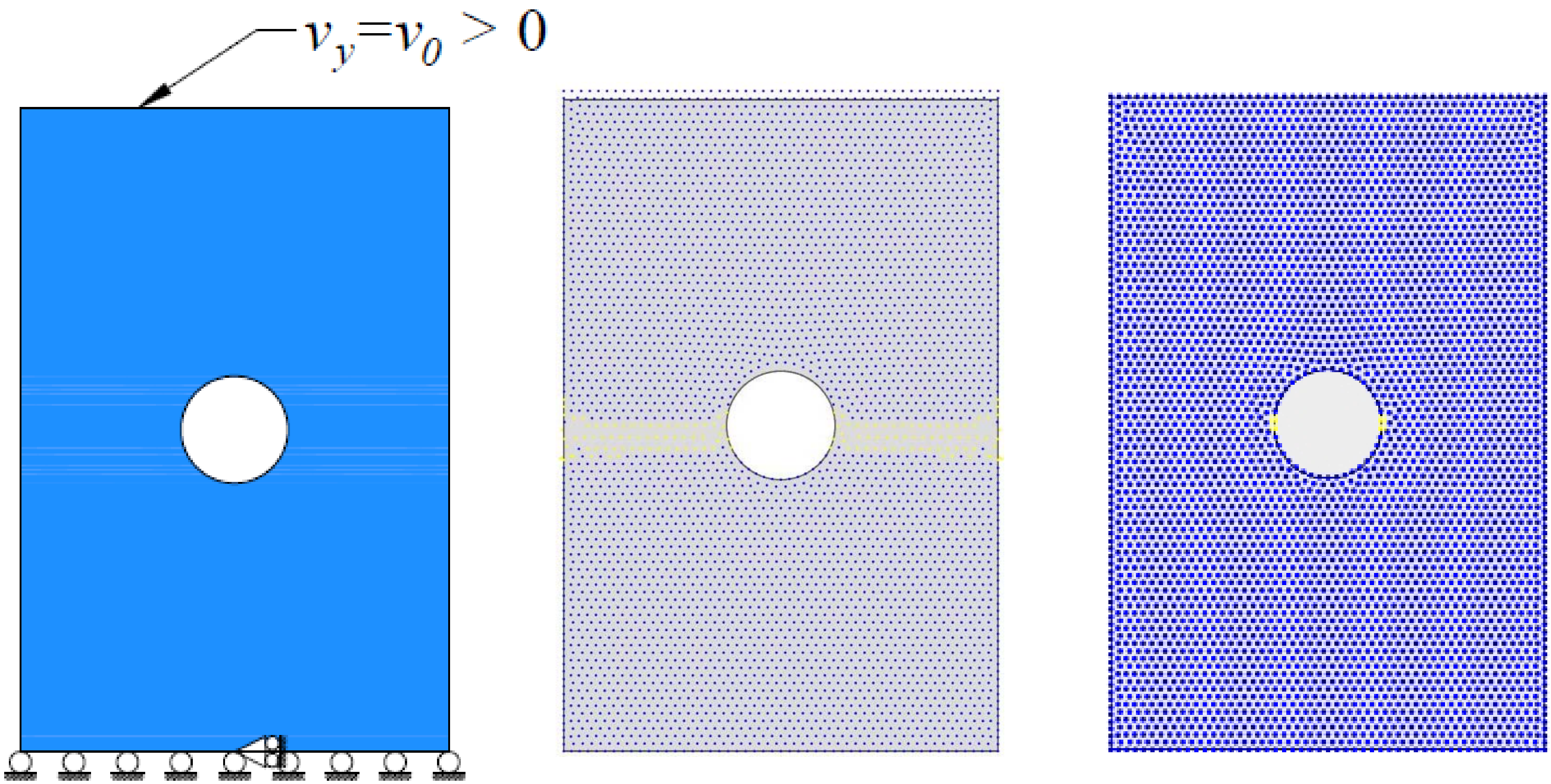


GraFEA: Dependence of nodal force on edge- strains



Khodabakhshi, P, Reddy, J.N., Srinivasa, A.R., 2016. GraFEA: a graph-based finite element approach for the study of damage and fracture in brittle materials, *Meccanica*, 51 (12): 3129 – 3147.

Capability of GraFEA to Study Fracture



LESSONS LEARNED (WORDS OF WISDOM)

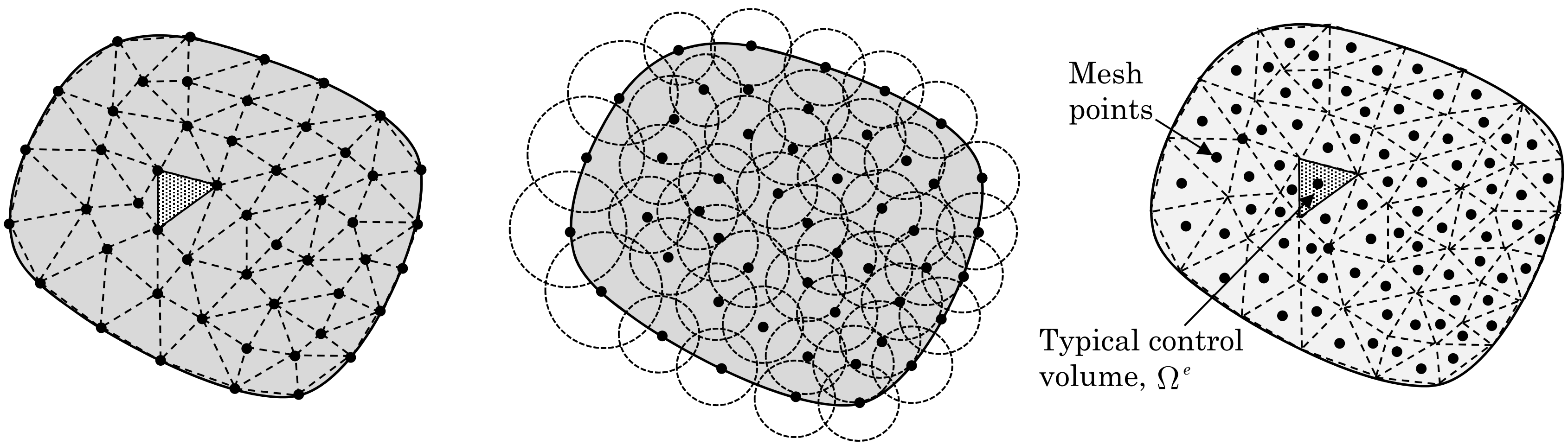
- Engineers and scientists "model" phenomena that occurs in nature.
- Continuum mechanics is a means to an end; that is, it provides tools to construct a mathematical model, analyze, and make a decision (towards designing and building).
- There is no "complete" or "exact" mathematical model of anything we like to model & analyze.
- We can only try to "improve" on what we already know (often, goal-based thinking).
- Only two things that matter in engineering: (1) Reliable functionality (or probability of failure) and (2) cost of the product.

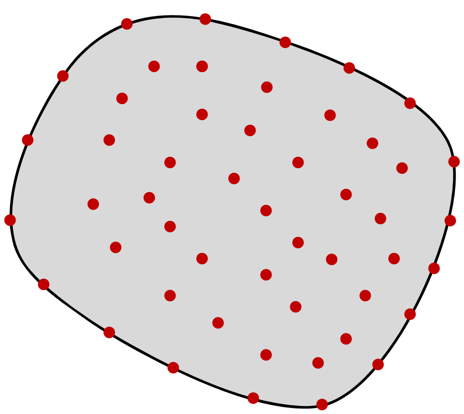
WORDS OF WISDOM (continued)

- **Differentiability of field variables is not an inherent attribute; we endowed them so that we can gain some insights without solving complex problems.**

- **With the computational tools we have, we can account for missing terms, or reformulate the classical continuum mechanics with non-classical continuum mechanics (e.g., strain-gradient theories, peridynamics, and others).**

MAIN IDEA BEHIND ALL DISCRETE METHODS





In the end, all numerical methods involve setting up algebraic relations between the values of the duality pairs (cause and effect) at selected points of the continuum.

CLOSING REMARKS

- There is experimental as well as modeling evidence that indicates the non-locality in materials manifests in different forms.
- Non-classical continuum mechanics brings additional means to address missing effects from the classical mechanics and explains certain essential mechanisms that are observed in experiments.
- Eringen's differential model is a diffusion type stress-gradient model. It shows stiffness reduction (flexibility) effect. Thus, it has limited application.

CLOSING REMARKS (continued)

- Strain gradient and modified coupe stress theories are related, and they show stiffening effect and allow for multiple length scales. They can be used to model large structures without using full 3-D models.
- Generalized (or non-classical) continuum theories are required to model material behavior more accurately. Such theories predict reduction in stress concentration factor around holes and cracks, which can give rise to improved toughness.
- GraFEA has a great potential and it needs to be developed further for inelastic and ductile materials.

SUMMARY REMARKS

- Our works must be built on sound mechanics foundation (wisdom to see details).
- We must develop robust computational tools that make use of advances made in theoretical developments and numerical methods.
- We must seek physically meaningful experimental validations to understand and predict the risks of failure (i.e., understand what is happening and use it to assess risk of failure).

I thank you
for your interest in my lecture

I thank
The Committee on
City U Distinguished Lecture Series
and
Professors C.W. Lim and Q.S. Li

That which is not given is lost