LECTURE III

WKB METHOD AND TURNING POINT

The transition from the classical physics of the late nineteenth century to the quantum mechanics of the early twentieth century is examplified by the problem of finding uniformly asymptotic solutions of the Liouville-Green (WKB) equation.

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1. Introduction

Differential equations of the type

$$y''(x) + \{\lambda^2 a(x) + b(x)\}y(x) = 0 \tag{1.1}$$

arise frequently in mathematical physics, where λ is a positive parameter. We first consider the simplest case, in which $a(x)$ is a real, positive, and twice continuously differentiable function in a given finite or infinite interval (a_1, a_2). We also assume that $b(x)$ is a continuous real- or complex-valued function. Let

$$\xi = \int a^{1/2}(x)dx, \qquad\qquad w = a^{1/4}(x)y(x). \tag{1.2}$$

It is readily verified that under this transformation, equation (1.1) becomes

$$\frac{d^2w}{d\xi^2} + \{\lambda^2 + \psi(\xi)\}w = 0, \tag{1.3}$$

where

$$\psi(\xi) = \frac{5}{16}\frac{a'^2(x)}{a^3(x)} - \frac{1}{4}\frac{a''(x)}{a^2(x)} + \frac{b(x)}{a(x)}. \tag{1.4}$$

The change of variables from (x, y) to (ξ, w) is known as the *Liouville transformation.*

If we discard ψ in (1.3), then we obtain two linearly independent solutions $e^{\pm i\lambda\xi}$. In terms of the original variables, we get

$$\begin{aligned} y(x) \sim & Aa^{-1/4}(x)\exp\{i\lambda\int a^{1/2}(x)dx\} \\ & + Ba^{-1/4}(x)\exp\{-i\lambda\int a^{1/2}(x)dx\}, \end{aligned} \tag{1.5}$$

where A and B are arbitrary constants. Equation (1.5) is known as the *Liouville-Green approximation*, whereas physicists refer to (1.5) as the *WKB* (or *semiclassical*) *approximation* in recognition of the work of Wentzel (1926),

Kramers (1926) and Brillouin (1926). The contribution of these authors was, however, not really the construction of approximation (1.5), but the connection of exponential and oscillatory approximations across a *turning point*, i.e., a zero of $a(x)$.

In Sec. 2, we will give a rigorous proof of (1.5) from which one will learn a basic argument frequently used to establish the validity of asymptotic solutions to differential equations. From this proof, we will also see a double asymptotic feature in the Liouville-Green approximation, that is, it sometimes holds either as $\lambda \to \infty$ with x fixed, or as $x \to \infty$ with λ fixed. In Sec. 3, we introduce the Langer transformation, and present a uniform asymptotic solution in the neighborhood of a turning point. Sec. 4 deals with the case in which the coefficient functions $a(x)$ and $b(x)$ in (1.1) have, respectively, a simple and a double pole in the interval (a_1, a_2). The final section contains several examples to illustrate the usefulness of the approximations obtained in the previous sections. Most of the material for this lecture is taken from the definitive book by Olver [8].

2. Successive Approximations

The most frequently used method to prove asymptotic results for differential equations is probably the method of successive approximations. In this section, we shall illustrate this method by establishing the validity of (1.5). In (1.3) we substitute

$$w(\xi) = e^{i\lambda\xi}[1 + h(\xi)], \tag{2.1}$$

and obtain

$$h''(\xi) + i2\lambda h'(\xi) = -\psi(\xi)[1 + h(\xi)]. \tag{2.2}$$

We view (2.2) as an inhomogeneous second-order differential equation in $h(\xi)$. By the principle of variation of parameters, one can convert (2.2) into the integral equation

$$h(\xi) = -\frac{1}{i2\lambda}\int_\alpha^\xi \{1 - e^{i2\lambda(v-\xi)}\}\psi(v)[1 + h(v)]dv, \tag{2.3}$$

where α is the value of ξ at $x = a, a = a_1$ or a_2, and we assume α is finite. One can easily verify that any solution of this integral equation is also a solution of the differential equation (2.2).

Define $h_0(\xi) = 0$ and

$$h_{s+1}(\xi) = -\frac{1}{i2\lambda}\int_\alpha^\xi \{1 - e^{i2\lambda(v-\xi)}\}\psi(v)[1 + h_s(v)]dv \tag{2.4}$$

for $s = 0, 1, 2, \cdots$. Since $|1 - e^{i2\lambda(v-\xi)}| \le 2$, we have $|h_1(\xi)| \le \frac{1}{\lambda}\Psi(\xi)$, where

$$\Psi(\xi) = \int_\alpha^\xi |\psi(v)|dv. \tag{2.5}$$

Suppose that for $s = k$, we have

$$|h_k(\xi) - h_{k-1}(\xi)| \leq \frac{\Psi^k(\xi)}{\lambda^k k!}. \tag{2.6}$$

Then, from (2.4) it follows that

$$h_{k+1}(\xi) - h_k(\xi) = -\frac{1}{i2\lambda}\int_\alpha^\xi \{1 - e^{i2\lambda(v-\xi)}\}\psi(v)[h_k(v) - h_{k-1}(v)]dv. \tag{2.7}$$

Hence

$$|h_{k+1}(\xi) - h_k(\xi)| \leq \frac{1}{\lambda^{k+1}k!}\int_\alpha^\xi |\psi(v)|\Psi^k(v)dv = \frac{\Psi^{k+1}(\xi)}{\lambda^{k+1}(k+1)!}.$$

By induction, (2.6) holds for all $k \geq 1$. Since $\Psi(\xi)$ is bounded when ξ is finite, the series

$$h(\xi) = \lim_{n\to\infty} h_n(\xi) = \sum_{k=0}^{\infty}\{h_{k+1}(\xi) - h_k(\xi)\} \tag{2.8}$$

is uniformly convergent on any compact ξ-interval. Taking the limit as $n \to \infty$ shows that $h(\xi)$ is a solution of the integral equation (2.3), and hence a solution of the second-order differential equation (2.2). To show that $h(\xi)$ is twice continuously differentiable, we note that (2.4) gives

$$h_1'(\xi) = -\int_\alpha^\xi e^{i2\lambda(v-\xi)}\psi(v)dv, \tag{2.9}$$

and we have from (2.7)

$$h_{k+1}'(\xi) - h_k'(s) = -\int_\alpha^\xi e^{i2\lambda(v-\xi)}\psi(v)[h_k(v) - h_{k-1}(v)]dv. \tag{2.10}$$

Again since $|1 - e^{i2\lambda(v-\xi)}| \leq 2$, by (2.6) we obtain

$$|h_{k+1}'(\xi) - h_k'(\xi)| \leq \frac{\Psi^{k+1}(\xi)}{\lambda^k(k+1)!}, \qquad k = 0, 1, 2, \cdots,$$

which clearly shows the uniform convergence of the series $\sum\{h_{k+1}'(\xi) - h_k'(\xi)\}$ in any compact interval. Furthermore, from (2.2) we have

$$h_{k+1}''(\xi) - h_k''(\xi) = -i2\lambda[h_{k+1}'(\xi) - h_k'(\xi)] - \psi(\xi)[h_k(\xi) - h_{k-1}(\xi)].$$

Uniform convergence of $\sum\{h_{k+1}''(\xi) - h_k''(\xi)\}$ is now evident, and $h(\xi)$ is twice continuously differentiable.

In summary, we have shown that equation (1.3) is satisfied by the function $w(\xi)$ in (2.1) with $h(\xi)$ given by (2.8). Summation of (2.6) gives

$$|h(\xi)| \leq \exp\left\{\frac{1}{\lambda}\Psi(\xi)\right\} - 1. \tag{2.11}$$

To express the result in terms of the original variables, we introduce the *control function*

$$F(x) = \int \left\{ \frac{1}{a^{1/4}} \frac{d^2}{dx^2} \left(\frac{1}{a^{1/4}} \right) + \frac{b}{a^{1/2}} \right\} dx \tag{2.12}$$

and the notation

$$\mathcal{V}_{a,x}(F) = \int_a^x |F'(t)| dt \tag{2.13}$$

for the total variation of F over the interval (a, x). It is readily verified that $\Psi(\xi) = \mathcal{V}_{a,x}(F)$. Hence, on account of (1.2), equation (2.1) can be written as

$$y_1(x) = a^{-1/4}(x) \exp\{i\lambda \int a^{1/2}(x) dx\}[1 + \varepsilon_1(\lambda, x)] \tag{2.14}$$

with

$$|\varepsilon_1(\lambda, x)| \leq \exp\left\{ \frac{1}{\lambda} \mathcal{V}_{a,x}(F) \right\} - 1. \tag{2.15}$$

The same argument will yield the second linearly independent solution

$$y_2(x) = a^{-1/4}(x) \exp\{-i\lambda \int a^{1/2}(x) dx\}[1 + \varepsilon_2(\lambda, x)], \tag{2.16}$$

where $\varepsilon_2(\lambda, x)$ also satisfies (2.15), i.e.,

$$|\varepsilon_2(\lambda, x)| \leq \exp\left\{ \frac{1}{\lambda} \mathcal{V}_{a,x}(F) \right\} - 1. \tag{2.17}$$

For fixed x and large λ, the right-hand sides of (2.15) and (2.17) are both $O(\lambda^{-1})$. Hence, a general solution to (1.1) has the asymptotic behavior given in (1.5).

In (2.15) and (2.17), we can take $a = a_1$ and $a = a_2$, respectively. If $\mathcal{V}_{a_1,a_2}(F) < \infty$, then the O-terms obtained from (2.15) and (2.17) are uniform with respect to x.

The corresponding result for the equation

$$y''(x) - \{\lambda^2 a(x) + b(x)\} y(x) = 0 \tag{2.18}$$

is that we have two linearly independent solutions of the form

$$y_1(x) = a^{-1/4}(x) \exp\{\lambda \int a^{1/2}(x) dx\}[1 + \varepsilon_1(\lambda, x)] \tag{2.19}$$

and

$$y_2(x) = a^{-1/4}(x) \exp\{-\lambda \int a^{1/2}(x) dx\}[1 + \varepsilon_2(\lambda, x)], \tag{2.20}$$

where

$$|\varepsilon_j(\lambda, x)| \le \exp\left\{\frac{1}{2\lambda}\mathcal{V}_{a_j,x}(F)\right\} - 1 \qquad (j = 1, 2). \tag{2.21}$$

If $\mathcal{V}_{a_1,a_2}(F) < \infty$, then (2.21) gives

$$y_j(x) = a^{-1/4}(x)\exp\{(-1)^{j-1}\int a^{1/2}(x)dx\}[1 + O(\lambda^{-1})], \tag{2.22}$$

$j = 1, 2$, which holds uniformly for $x \in (a_1, a_2)$.

The proof of (2.21) is very similar to that of (2.15). The integral equation corresponding to (2.3) is

$$h(\xi) = -\frac{1}{2\lambda}\int_{\alpha_1}^{\xi}\{1 - e^{2\lambda(v-\xi)}\}\psi(v)[1 + h(v)]dv, \tag{2.23}$$

where $\xi = \alpha_1$ corresponds to $x = a_1$. Since $\xi - v \ge 0$, instead of the bound $|1 - e^{i2\lambda(v-\xi)}| \le 2$ used in proving (2.6), we can now use the estimate

$$0 \le 1 - e^{2\lambda(v-\xi)} < 1. \tag{2.24}$$

As a result, we have the extra factor $\frac{1}{2}$ in the total variation $\mathcal{V}_{a_1,x}(F)$ in (2.21). The result for $j = 2$ follows by replacing x in (2.18) by $-x$.

It is interesting to note that the error bounds in (2.21) can also be used to give asymptotic properties of the approximations (2.19) and (2.20) in the neighborhood of a singularity of the differential equation. Because of this *double* asymptotic feature, the Liouville-Green approximation is indeed a remarkably powerful tool for approximating solutions of linear second-order differential equations. To illustrate our point, we let $\lambda = 1$ in (2.18), and assume that $\mathcal{V}_{a_1,a_2}(F) < \infty$ and also

$$\int a^{1/2}(x)dx \to \infty \qquad \text{as} \quad x \to a_2^-. \tag{2.25}$$

Under these conditions, we shall show that the error term $\varepsilon_1(x) := \varepsilon_1(1, x)$ in (2.19) satisfies

$$\varepsilon_1(x) \to \text{ a constant} \qquad \text{as } x \to a_2^-. \tag{2.26}$$

This together with (2.19) shows that there is a solution $y_3(x)$ such that

$$y_3(x) \sim a^{-1/4}(x)\exp\left\{\int a^{1/2}(x)dx\right\}, \qquad x \to a_2^-. \tag{2.27}$$

Coupling (2.20) and (2.27), we obtain two linearly independent asymptotic solutions as $x \to a_2^-$.

We now proceed to prove (2.26). From (2.21), we know that $\varepsilon_1(x)$ is bounded in (a_1, a_2). What (2.26) says is that $\varepsilon_1(x)$ does not oscillate infinitely

often as $x \to a_2$. In view of (2.25), we have $\alpha_2 = \infty$ by (1.2). Since $\mathcal{V}_{a_1,a_2}(F) < \infty$, for any given $\varepsilon > 0$, there exists a positive number $\delta \in (\alpha_1, \infty)$ such that

$$\int_\delta^\infty |\psi(v)|dv = \varepsilon.$$

Now assume that $\xi \geq \delta$, and subdivide the interval of integration in (2.9) at δ. Note that in the present case, $\lambda = 1$ and "i" is absent; cf. (2.23). Thus, we have

$$|h_1'(\xi)| \leq \int_{\alpha_1}^\delta e^{2(v-\xi)}|\psi(v)|dv + \int_\delta^\xi |\psi(v)|dv \leq e^{2(\delta-\xi)}\Psi(\delta) + \varepsilon.$$

Similarly, from (2.10) and (2.6) we obtain

$$|h_{k+1}'(\xi) - h_k'(\xi)| \leq \frac{e^{2(\delta-\xi)}\Psi^{k+1}(\delta)}{(k+1)!\,2^k} + \frac{\Psi^k(\infty)}{k!\,2^k}\,\varepsilon$$

for $k \geq 1$. The extra factor of 2^{-k} comes from the fact that we use the estimates in (2.24) for the case of eqution (2.18), instead of the bound $|1 - e^{i2\lambda(v-\xi)}| \leq 2$ for the case of equation (1.1). From (2.8), it follows that

$$|h'(\xi)| \leq 2e^{2(\delta-\xi)}\{e^{\Psi(\delta)/2} - 1\} + e^{\Psi(\infty)/2}\,\varepsilon. \tag{2.28}$$

The first term on the right-hand side tends to zero as $\xi \to \infty$, and since ε is arbitrary, (2.28) implies that $h'(\xi) \to 0$ as $\xi \to \infty$.

With $\lambda = 1$ in (2.23), it can be verified by differentiation that

$$h(\xi) = -\frac{1}{2}h'(\xi) - \frac{1}{2}\int_{\alpha_1}^\xi \psi(v)[1 + h(v)]dv. \tag{2.29}$$

If we let

$$l_0(\xi) = \int_{\alpha_1}^\xi \psi(v)dv, \qquad l_k(\xi) = \int_{\alpha_1}^\xi \psi(v)\{h_k(v) - h_{k-1}(v)\}dv \qquad (k \geq 1),$$

then in view of (2.8) we can rewrite (2.29) as

$$h(\xi) = -\frac{1}{2}h'(\xi) - \frac{1}{2}\sum_{k=0}^\infty l_k(\xi). \tag{2.30}$$

For $\xi \geq \delta$, we have from (2.6)

$$|l_k(\xi) - l_k(\delta)| \leq \frac{\Psi^k(\xi)}{k!\,2^k}\int_\delta^\xi |\psi(v)|dv, \qquad k \geq 0, \tag{2.31}$$

since $\Psi(\xi)$ is an increasing function. Coupling (2.30) and (2.31) gives

$$|h(\xi) - h(\delta)| \leq \frac{1}{2}e^{\Psi(\xi)/2}\{\Psi(\xi) - \Psi(\delta)\} + \frac{1}{2}|h'(\xi) - h'(\delta)|.$$

The right-hand side tends to zero as ξ and δ approach infinity independently. Hence, $h(\xi)$ must tend to a constant as $\xi \to \infty$; this completes the proof of (2.26).

3. Turning Point Problem

The problem of finding asymptotic solutions to the differential equation (2.18) becomes much more complicated, when the coefficient function $a(x)$ has a zero, say at $x = x_0$, in the interval (a_1, a_2). Such a point is known as a *turning point* of the differential equation. In this case, there is an ambiguity in taking the square root of the function $a(x)$, and hence the Liouville transformation (1.2) is not well-defined.

For definiteness, we assume that $a(x)$ has the same sign as $x - x_0$; i.e.,

$$a(x)(x - x_0) > 0 \qquad \text{for all} \quad x \neq x_0. \tag{3.1}$$

Instead of (1.2), we now make the change of variables

$$\begin{cases} \dfrac{2}{3}\zeta^{3/2} = \int_{x_0}^{x} a(t)^{1/2} dt, & x_0 \leq x, \\ \dfrac{2}{3}(-\zeta)^{3/2} = \int_{x}^{x_0} [-a(t)]^{1/2} dt, & x < x_0, \end{cases} \tag{3.2}$$

and

$$w(\zeta) = \left(\frac{d\zeta}{dx}\right)^{1/2} y. \tag{3.3}$$

It is easily verified that

$$\left(\frac{d\zeta}{dx}\right)^2 = \frac{a(x)}{\zeta}. \tag{3.4}$$

The transformation $(x, y) \mapsto (\zeta, w)$ was first introduced by Langer [3], under which equation (1.1) becomes

$$\frac{d^2 w}{d\zeta^2} = \{\lambda^2 \zeta + \psi(\zeta)\} w, \tag{3.5}$$

where

$$\psi(\zeta) = \frac{5}{16}\zeta^{-2} + \{4a(x)a''(x) - 5[a'(x)]^2\} \frac{\zeta}{16\, a^3(x)} + \frac{\zeta b(x)}{a(x)}. \tag{3.6}$$

If ψ in (3.5) is neglected, then we have the Airy equation

$$\frac{d^2 w}{d\zeta^2} = \lambda^2 \zeta w,$$

two linearly independent solutions of which are the Airy functions $\mathrm{Ai}(\lambda^{2/3}\zeta)$ and $\mathrm{Bi}(\lambda^{2/3}\zeta)$. Using the method of successive approximation, one can show,

as in Sec. 2, that equation (3.5) has twice continuously differentiable solutions given by

$$\begin{aligned} y_1(x) &= \hat{a}^{-1/4}(x)[\mathrm{Ai}(\lambda^{2/3}\zeta) + \varepsilon_1(\lambda, x)], \\ y_2(x) &= \hat{a}^{-1/4}(x)[\mathrm{Bi}(\lambda^{2/3}\zeta) + \varepsilon_2(\lambda, x)], \end{aligned} \tag{3.7}$$

where $\hat{a}(x) = a(x)/\zeta$; see (3.3) and (3.4). To give an estimate for the error terms $\varepsilon_1(\lambda, x)$ and $\varepsilon_2(\lambda, x)$, we first introduce the error-control function

$$H(x) := -\int_0^{\zeta} |v|^{-1/2}\psi(v)dv. \tag{3.8}$$

In terms of the original variable, it is equivalent to

$$H(x) = \int_{x_0}^{x} \left\{ \frac{1}{|a|^{1/4}} \frac{d^2}{dx^2}\left(\frac{1}{|a|^{1/4}}\right) - \frac{b}{|a|^{1/2}} - \frac{5|a|^{1/2}}{16|\zeta|^3} \right\} dx. \tag{3.9}$$

The *modulus function* $M(x)$ and the *weight function* $E(x)$ associated with the Airy functions $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$ are defined by $E(x) = 1$ for $-\infty < x \leq c$,

$$E(x) = \{\mathrm{Bi}(x)/\mathrm{Ai}(x)\}^{1/2}, \qquad c \leq x < \infty, \tag{3.10}$$

and

$$M(x) = \{E^2(x)\mathrm{Ai}^2(x) + E^{-2}(x)\mathrm{Bi}^2(x)\}^{1/2}, \tag{3.11}$$

where $E^{-1}(x) = 1/E(x)$ and c denotes the negative root of the equation

$$\mathrm{Ai}(x) = \mathrm{Bi}(x)$$

with smallest absolute value. Numerical calculation has shown that $c = -0.36605$, correct to five decimal places. The modulus and the weight have the well known asymptotic behavior

$$M(x) \sim \pi^{-1/2}|x|^{-1/4}, \qquad x \to \pm\infty, \tag{3.12}$$

and

$$E(x) \sim 2^{1/2}\exp\left(\frac{2}{3}x^{3/2}\right), \qquad x \to +\infty. \tag{3.13}$$

The error terms $\varepsilon_1(\lambda, x)$ and $\varepsilon_2(\lambda, x)$ in (3.7) satisfy

$$\frac{|\varepsilon_1(\lambda, x)|}{M(\lambda^{2/3}\zeta)} \leq \frac{E^{-1}(\lambda^{2/3}\zeta)}{\mu}\left\{\exp\left[\frac{\mu\mathcal{V}_{x,a_2}(H)}{\lambda}\right] - 1\right\} \tag{3.14}$$

and

$$\frac{|\varepsilon_2(\lambda, x)|}{M(\lambda^{2/3}\zeta)} \leq \frac{E(\lambda^{2/3}\zeta)}{\mu}\left\{\exp\left[\frac{\mu\mathcal{V}_{a_1,x}(H)}{\lambda}\right] - 1\right\}, \tag{3.15}$$

where

$$\mu = \sup_{(-\infty,\infty)} \{\pi|x|^{1/2}M^2(x)\} = 1.04\cdots. \tag{3.16}$$

The above method can be extended to the more general equation

$$y''(x) = \lambda^2 a(\lambda, x)y(x), \tag{3.17}$$

where $a(\lambda, x)$ has an asymptotic expansion

$$a(\lambda, x) \sim a_0(x) + \frac{a_1(x)}{\lambda} + \frac{a_2(x)}{\lambda^2} + \cdots \tag{3.18}$$

as $|\lambda| \to \infty$, and $a_0(x)$ has a zero x_0 in the interval (a_1, a_2). Applying the Langer transformation (3.2) – (3.3) with $a(x)$ replaced by $a_0(x)$, equation (3.17) becomes

$$\frac{d^2w}{d\zeta^2} = \{\lambda^2\zeta + \lambda\phi(\zeta) + \psi(\lambda, \zeta)\}w, \tag{3.19}$$

where we put $\hat{a}(x) = a_0(x)/\zeta$,

$$\phi(\zeta) = a_1(x)/\hat{a}(x) \tag{3.20}$$

and

$$\psi(\lambda, \zeta) \sim \sum_{s=0}^{\infty} \frac{\psi_s(\zeta)}{\lambda^s}, \tag{3.21}$$

with

$$\psi_0(\zeta) = \frac{a_2(\zeta)}{\hat{a}(\zeta)} - \frac{1}{\hat{a}^{3/4}(\zeta)}\frac{d^2}{d\zeta^2}\left\{\frac{1}{\hat{a}^{1/4}(x)}\right\} \tag{3.22}$$

and $\psi_s(\zeta) = a_{s+2}(\zeta)/\hat{a}(\zeta)$ for $s \geq 1$. An asymptotic series solution to (3.19) is given by

$$\begin{aligned} w(\zeta) \sim & \mathrm{Ai}\left(\lambda^{2/3}\zeta + \frac{\Phi}{\lambda^{1/3}}\right)\sum_{s=0}^{\infty}\frac{A_s(\zeta)}{\lambda^s} \\ & + \frac{1}{\lambda^{4/3}}\mathrm{Ai}'\left(\lambda^{2/3}\zeta + \frac{\Phi}{\lambda^{1/3}}\right)\sum_{s=0}^{\infty}\frac{B_s(\zeta)}{\lambda^s}, \end{aligned} \tag{3.23}$$

where $\Phi = \Phi(\zeta)$ is defined by

$$\Phi(\zeta) = \frac{1}{2\zeta^{1/2}}\int_0^{\zeta}\frac{\phi(v)}{v^{1/2}}dv. \tag{3.24}$$

4. Simple Pole

Returning to equation (2.18), namely

$$y''(x) = \{\lambda^2 a(x) + b(x)\}y(x), \tag{4.1}$$

we now assume that $a(x)$ has a simple pole (say) at x_0 and $(x-x_0)^2 b(x)$ is analytic. For simplicity, we also assume that $a(x)$ has the same sign as $x-x_0$. In this case, Olver [7] has introduced the transformation

$$\begin{cases} \zeta^{1/2} = \displaystyle\int_{x_0}^{x} a^{1/2}(t)dt, & x \geq x_0, \\ (-\zeta)^{1/2} = \displaystyle\int_{x}^{x_0} \{-a(t)\}^{1/2} dt, & x \leq x_0, \end{cases} \tag{4.2}$$

and

$$w = \left(\frac{d\zeta}{dx}\right)^{1/2} y \tag{4.3}$$

which transforms (4.1) into the new equation

$$\frac{d^2 w}{d\zeta^2} = \left\{\frac{\lambda^2}{4\zeta} + \hat{\psi}(\zeta)\right\} w, \tag{4.4}$$

where

$$\hat{\psi}(\zeta) = \frac{b(x)}{\hat{a}(x)} + \frac{1}{\hat{a}^{1/4}(x)} \frac{d^2}{d\zeta^2}\{\hat{a}^{1/4}(x)\} \tag{4.5}$$

and $\hat{a}(x) = (d\zeta/dx)^2 = 4\zeta a(x)$. If $b(x)$ has a simple or double pole at x_0, then $\hat{\psi}(\zeta)$ has the same kind of singularity at $\zeta = 0$. Denote the value of $\zeta^2\hat{\psi}(\zeta)$ at $\zeta = 0$ by $\frac{1}{4}(\nu^2 - 1)$, and write (4.4) in the form

$$\frac{d^2 w}{d\zeta^2} = \left\{\frac{\lambda^2}{4\zeta} + \frac{\nu^2 - 1}{4\zeta^2} + \frac{\psi(\zeta)}{\zeta}\right\} w \tag{4.6}$$

with $\psi(\zeta) = \zeta\hat{\psi}(\zeta) - \frac{1}{4}(\nu^2 - 1)\zeta^{-1}$. Note that $\psi(\zeta)$ is analytic at $\zeta = 0$. In terms of the original variable, $\frac{1}{4}(\nu^2 - 1)$ is the value of $(x-x_0)^2 b(x)$ at $x = x_0$, and

$$\psi(\zeta) = \frac{1 - 4\nu^2}{16\zeta} + \frac{b(x)}{4a(x)} + \frac{4a(x)a''(x) - 5a'^2(x)}{64a^3(x)}. \tag{4.7}$$

The differential equation (4.6) has a regular singularity at $\zeta = 0$, and we suppose that the range of ζ is a real interval (α_1, α_2) which contains $\zeta = 0$ and may be unbounded. We consider separately the intervals $[0, \alpha_2)$ and $(\alpha_1, 0]$.

If the term $\psi(\zeta)/\zeta$ is neglected, then (4.6) becomes

$$\frac{d^2 w}{d\zeta^2} = \left\{\frac{\lambda^2}{4\zeta} + \frac{\nu^2 - 1}{4\zeta^2}\right\} w. \tag{4.8}$$

Two linearly independent solutions of (4.8) are $\zeta^{1/2} I_\nu(\lambda\zeta^{1/2})$ and $\zeta^{1/2} K_\nu(\lambda\zeta^{1/2})$. It can be demonstrated, as in the previous sections, that if $\zeta^{-1/2}\psi(\zeta)$ is absolutely integrable on $[0, \alpha_2)$, then equation (4.6) has two linearly independent solutions $w_1(\lambda, \zeta)$ and $w_2(\lambda, \zeta)$ such that

$$w_1(\lambda, \zeta) = \zeta^{1/2} I_\nu(\lambda\zeta^{1/2})[1 + O(\lambda^{-1})], \tag{4.9}$$

$$w_2(\lambda,\zeta) = \zeta^{1/2} K_\nu(\lambda\zeta^{1/2})[1 + O(\lambda^{-1})], \tag{4.10}$$

as $|\lambda| \to \infty$, where the O-terms hold uniformly with respect to $\zeta \in [0, \alpha_2)$.

When ζ is negative, two linearly independent solutions of the reduced equation (4.8) are $|\zeta|^{1/2} J_\nu(\lambda|\zeta|^{1/2})$ and $|\zeta|^{1/2} Y_\nu(\lambda|\zeta|^{1/2})$. Hence, if $\zeta^{-1/2}\psi(\zeta)$ is absolutely integrable on $(\alpha_1, 0]$, equation (4.6) has two solutions

$$w_1(\lambda,\zeta) = |\zeta|^{1/2} J_\nu(\lambda|\zeta|^{1/2})[1 + O(\lambda^{-1})], \tag{4.11}$$

$$w_2(\lambda,\zeta) = |\zeta|^{1/2} Y_\nu(\lambda|\zeta|^{1/2})[1 + O(\lambda^{-1})], \tag{4.12}$$

as $|\lambda| \to \infty$, which hold uniformly for $\zeta \in (\alpha_1, 0]$.

5. Examples

As an illustration of the results obtained in Secs. 2-4, here we give three concrete examples. The first one demonstrates that the Liouville-Green (WKB) approximation is indeed a powerful tool for approximating solutions of linear second-order differential equations. The second one provides a uniform asymptotic approximation for the polynomials orthogonal with respect to the weight $\exp(-x^4)$ on the real line. The final example deals with the Jacobi function $\varphi_\mu^{(\alpha,\beta)}(t), t > 0$.

Example 1. In an interesting paper [9] on a very abstract topic (namely, Hardy fields), M. Rosenlicht considered solutions of the equation

$$y''(x) = x^x y(x) \tag{5.1}$$

as $x \to \infty$. In view of the rapid growth of the coefficient function x^x as $x \to \infty$, it is really not easy to guess the large x-behavior of the solution $y(x)$. Let us try the results in Sec. 2, and choose $\lambda = 1, a(x) = x^x$ and $b(x) = 0$ in (2.18). The control function $F(x)$ in (2.12) is given by

$$F(x) = \frac{1}{16}\int [(1 + \ln x)^2 x^{-x/2} - 4x^{-(1+x/2)}]dx.$$

Clearly, $\mathcal{V}_{x,\infty}(F) \to 0$ as $x \to \infty$. Hence, (2.20) gives the recessive solution

$$y_2(x) \sim x^{-x/4} \exp\left(-\int x^{x/2} dx\right), \qquad x \to \infty. \tag{5.2}$$

A dominant solution is provided by (2.27), namely,

$$y_3(x) \sim x^{-x/4} \exp\left(\int x^{x/2} dx\right), \qquad x \to \infty. \tag{5.3}$$

An asymptotic expansion of the integral $\int x^{x/2} dx$ can be obtained by integration by parts.

Example 2. In [6], Nevai has studied the asymptotic behavior of the orthogonal polynomials

$$p_n(x) = \gamma_n x^n + \gamma_{n-1} x^{n-1} + \cdots, \qquad \gamma_n > 0, \tag{5.4}$$

associated with the weight function $\exp(-x^4)$ on the real line $\mathbb{R}$. These polynomials satisfy the recurrence relation

$$x p_n(x) = a_{n+1} p_{n+1}(x) + a_n p_{n-1}(x), \qquad n = 0, 1, \cdots, \tag{5.5}$$

with $p_0(x) = \gamma_0 > 0$ and $p_1(x) = \gamma_0 x / a_1$. The coefficients a_n are determined successively from the equation

$$n = 4a_n^2(a_{n+1}^2 + a_n^2 + a_{n-1}^2), \qquad n = 1, 2, \cdots, \tag{5.6}$$

where $a_0^2 = 0$ and $a_1^2 = \Gamma(\frac{3}{4})/\Gamma(\frac{1}{4})$. A two-term asymptotic expansion for a_n has been given by Lew and Quarles [4]. They showed that

$$a_n^2 = \left(\frac{n}{12}\right)^{1/2} \left[1 + \frac{1}{24n^2} + O\left(\frac{1}{n^4}\right)\right], \qquad n \to \infty. \tag{5.7}$$

If we let

$$\phi_n(x) = a_{n+1}^2 + a_n^2 + x^2, \tag{5.8}$$

then Shohat [10] and Nevai [5] independently showed that the function

$$z(x) = p_n(x)[\phi_n(x)]^{-1/2} \exp\left(-\frac{x^4}{2}\right) \tag{5.9}$$

satisfies the differential equation

$$z'' + f(n, x) z = 0, \tag{5.10}$$

where

$$\begin{aligned} f(n, x) = {} & 4a_n^2[4\phi_n(x)\phi_{n-1}(x) + 1 - 4a_n^2 x^2 - 4x^4 - 2x^2\phi_n(x)^{-1}] \\ & - 4x^6 - 4x^4\phi_n(x)^{-1} - 3x^2\phi_n(x)^{-1} + 6x^2 + \phi_n(x)^{-1}. \end{aligned} \tag{5.11}$$

If we make the change of variable

$$x = \lambda^{1/4} w \qquad \text{with} \qquad \lambda = \frac{4n}{3}, \tag{5.12}$$

then equation (5.10) becomes

$$\frac{d^2 z}{dw^2} = \lambda^2 \left[a_0(w) + \frac{a_1(w)}{\lambda} + \frac{a_2(w)}{\lambda^2} + \cdots\right] z, \tag{5.13}$$

where $a_0(w) = (4w^6 - 3w^2 - 1) = (2w^2+1)^2(w^2-1), a_1(w) = -(1+2w^2)$ and

$$a_2(w) = -\frac{20w^4 - 64w^2 + 17}{9(1+2w^2)^2}. \tag{5.14}$$

Since $a_0(w)$ vanishes at $w = \pm 1$, we have exactly the extended form of the turning point problem mentioned in (3.17) – (3.18), and the result (3.23) can be applied. The details of this study is given in [1].

Example 3. Let α, β and μ be real numbers with $\mu > 0$ and $\alpha \neq -1, -2, \cdots$. The Jacobi function is defined by

$$\varphi_\mu^{(\alpha,\beta)}(t) = {}_2F_1\left[\frac{1}{2}(\alpha+\beta+1-i\mu), \frac{1}{2}(\alpha+\beta+1+i\mu); \alpha+1; -\sinh^2 t\right] \tag{5.15}$$

for $t > 0$, where ${}_2F_1(a,b;c;z)$ is the Gaussian hypergeometric function. This function is related to the Jacobi polynomial

$$\frac{\Gamma(\alpha+1)\Gamma(n+1)}{\Gamma(\alpha+n+1)} P_n^{(\alpha,\beta)}(x) = {}_2F_1\left[-n, n+\alpha+\beta+1; \alpha+1; \frac{1}{2}(1-x)\right]. \tag{5.16}$$

The last formula furnishes the extension of the polynomial $P_n^{(\alpha,\beta)}(x)$ to arbitrary values of the degree n. From (5.15) and (5.16), it is evident that

$$\varphi_\mu^{(\alpha,\beta)}(t) = \frac{\Gamma(\alpha+1)\Gamma\big(\frac{1}{2}(i\mu-\alpha-\beta+1)\big)}{\Gamma\big(\frac{1}{2}(i\mu+\alpha-\beta+1)\big)} P^{(\alpha,\beta)}_{\frac{1}{2}(i\mu-\alpha-\beta-1)}(\cosh 2t), \tag{5.17}$$

and for this reason, $\varphi_\mu^{(\alpha,\beta)}(t)$ is called the Jacobi function.

It is known that $\varphi_\mu^{(\alpha,\beta)}(t)$ is the unique, even, C^∞-function on $\mathbb{R}$ which satisfies

$$\begin{aligned} v''(t) + [(2\alpha+1)\coth t + (2\beta+1)\tanh t]v'(t) \\ + [\mu^2 + (\alpha+\beta+1)^2]v(t) = 0 \end{aligned} \tag{5.18}$$

and $v(0) = 1$; see [2, p.2]. If we set

$$u(t) = (\sinh t)^{\alpha+\frac{1}{2}}(\cosh t)^{\beta+\frac{1}{2}}\varphi_\mu^{(\alpha,\beta)}(t), \tag{5.19}$$

then it is easily verified that

$$u''(t) + \left\{\mu^2 + \frac{\frac{1}{4}-\alpha^2}{\sinh^2 t} - \frac{\frac{1}{4}-\beta^2}{\cosh^2 t}\right\}u(t) = 0. \tag{5.20}$$

When $\alpha > -\frac{1}{2}$, we also have $u(0) = 0$.

To apply the asymptotic theory of Olver discussed in Sec. 4, we restrict ourselves to the case $\alpha \geq 0$ and introduce the new variables

$$(-\zeta)^{\frac{1}{2}} = t, \qquad \zeta < 0; \qquad W(\zeta) = (-\zeta)^{\frac{1}{4}}u(t). \tag{5.21}$$

The transformed equation is given by

$$\frac{d^2W}{d\zeta^2} = \left\{ \frac{\mu^2}{4\zeta} + \frac{\alpha^2-1}{4\zeta^2} + \frac{\psi(\zeta)}{\zeta} \right\} W(\zeta), \qquad \zeta < 0, \tag{5.22}$$

where

$$\psi(\zeta) = \frac{1}{4}\left\{ \frac{\frac{1}{4}-\alpha^2}{\zeta} + \left[\frac{\frac{1}{4}-\alpha^2}{\sinh^2(-\zeta)^{\frac{1}{2}}} - \frac{\frac{1}{4}-\beta^2}{\cosh^2(-\zeta)^{\frac{1}{2}}} \right] \right\}. \tag{5.23}$$

Note that $\psi(\zeta)$ is analytic at $\zeta = 0$.

For negative ζ, equations (4.11) and (4.12) give two asymptotic solutions to (5.22), one involving the Bessel function $J_\alpha(\mu\sqrt{-\zeta})$ and the other involving $Y_\alpha(\mu\sqrt{-\zeta})$. To identify the function $(-\zeta)^{\frac{1}{4}}u(t)$ in (5.21) with one of these two solutions or a linear combination of them, we note that from (5.19) and (5.21) we have

$$(-\zeta)^{\frac{1}{4}}u(t) \sim (-\zeta)^{(\alpha+1)/2}, \qquad \zeta \to 0^-. \tag{5.24}$$

Since $J_\alpha(x) \sim (x/2)^\alpha/\Gamma(\alpha+1)$ for x near zero, it follows from (4.11) and (5.21) that

$$(\sinh t)^{\alpha+\frac{1}{2}}(\cosh t)^{\beta+\frac{1}{2}}\varphi_\mu^{(\alpha,\beta)}(t) \sim \frac{2^\alpha\Gamma(\alpha+1)}{\mu^\alpha} t^{\frac{1}{2}} J_\alpha(\mu t); \tag{5.25}$$

for details, see [11].

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