LECTURE IV

BOUNDARY LAYER THEORY

The paper will certainly prove to be one of the most extraordinary papers of this century, and probably of many centuries

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1. Introduction

At the third International Congress of Mathematicians (1904) in Heidelberg, Ludwig Prandtl [16] presented a short paper entitled *On fluid motion with small frictions.* In 1972, at the *Symposium on the Future of Applied Mathematics*, George Carrier [3] made the remark that "This success is probably most surprising to rigor-oriented mathematicians (or applied mathematicians) when they realize that there still exists no theorem which speaks to the validity or the accuracy of Prandtl's treatment of his boundary-layer problem; but seventy years of observational experience leave little doubt of its validity and its value". With comments like these, one can readily appreciate the important position of Prandtl's principle of boundary-layer theory in the minds of applied mathematicians.

To illustrate Prandtl's idea, K.O. Friedrichs used the simple example

$$\begin{cases} \varepsilon \dfrac{d^2u}{dx^2} + \dfrac{du}{dx} = a + 2bx \\ u(0) = 0, \qquad u(1) = 1, \end{cases} \tag{1.1}$$

in his 1942 lecture notes [5]. Friedrichs' analysis can be easily extended to the more general singularly perturbed two-point boundary-value problem

$$\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = 0, \tag{1.2a}$$

$$y(-1) = A, \qquad\qquad y(1) = B. \tag{1.2b}$$

It is now well known that if $a(x)$ is positive, then the asymptotic solution which holds uniformly in the interval $[-1, 1]$ is given by

$$\begin{aligned} y_{\text{unif}}(x) = B \exp\left(\int_x^1 \frac{b(t)}{a(t)} dt\right) \\ + \left\{A - B \exp\left(\int_{-1}^1 \frac{b(t)}{a(t)} dt\right)\right\} e^{-a(-1)(1+x)/\varepsilon}. \end{aligned} \tag{1.3}$$

This formula is provided in at least eight standard texts; see, e.g., [2, p.425], [6, pp.53 & 58], [7, p.59], [9, p.68], [10, p.421], [11, p.289], [13, p.94] and [17, p.109]. Despite what Carrier had said above, the lack of rigor in Prandtl's

boundary-layer theory does raise some concern from mathematicians who believe that arguments based on purely heuristic reasoning may lead to incorrect results.

A derivation of equation (1.3) is given in Sec. 2, where we also point out that care must be taken in the use of (1.3) when exponentially small terms are involved. In Sec. 3, we consider a case in which the coefficient function $a(x)$ in (1.2a) has a zero. More precisely, we discuss the case when

$$a(x) \sim \alpha x \qquad \text{and} \qquad b(x) \sim \beta, \qquad \text{as } x \to 0, \tag{1.4}$$

where $\alpha \neq 0$ and β are constants. Our discussion will be divided into three subcases; namely, (i) $\alpha > 0$ and $\beta/\alpha \neq 1, 2, \cdots$; (ii) $\alpha < 0$ and $\beta/\alpha \neq 0, -1, -2, \cdots$; (iii) $\alpha > 0$ and $\beta/\alpha = 1, 2, \cdots$, or $\alpha < 0$ and $\beta/\alpha = 0, -1, -2, \cdots$. In case (i), as we shall see, the solution has an internal-layer behavior. The so-called Ackerberg-O'Malley reasonance refers to case (iii). Sec. 4 is devoted to the nonlinear equation

$$\varepsilon u'' + u^2 = 1, \qquad -1 < x < 1, \tag{1.5}$$

with boundary conditions

$$u(-1) = u(1) = 0. \tag{1.6}$$

2. Derivation of (1.3)

Since ε is small, it is natural to set $\varepsilon = 0$ in (1.2a) so that we obtain the *reduced* equation

$$a(x)Y'(x) + b(x)Y(x) = 0. \tag{2.1}$$

The general solution of this equation is

$$Y(x) = K \exp\left(\int_x^1 \frac{b(t)}{a(t)} dt\right),$$

K being an arbitrary constant. The boundary condition at $x = 1$ immediately suggests that $K = B$ and

$$Y(x) = B \exp\left(\int_x^1 \frac{b(t)}{a(t)} dt\right). \tag{2.2}$$

In general, $Y(x)$ can not satisfy the boundary condition at $x = -1$. Hence, the approximate solution $Y(x)$ is valid only in an interval near $x = 1$. This interval is known as the *outer region*, and $Y(x)$ is called the *outer solution*. In the interval near $x = -1$, which is known as the *boundary-layer* (or *inner*) *region*, we make the change of variable $\xi = \frac{x+1}{\varepsilon}$ and define

$$y(x) = y(\varepsilon\xi - 1) \equiv \overline{y}(\xi).$$

Clearly, $\overline{y}(\xi)$ satisfies the new equation

$$\overline{y}''(\xi) + a(\varepsilon\xi - 1)\overline{y}'(\xi) + \varepsilon b(\varepsilon\xi - 1)\overline{y}(\xi) = 0.$$

Setting $\varepsilon = 0$ gives another simplified equation

$$y''_{BL}(\xi) + a(-1)y'_{BL}(\xi) = 0. \tag{2.3}$$

This equation can again be solved explicitly, and the general solution is given by

$$y_{BL}(\xi) = C_1 + C_2 e^{-a(-1)\xi},$$

where C_1 and C_2 are arbitrary constants. We call $y_{BL}(\xi)$ the *boundary-layer* (or *inner*) solution. The boundary condition at $x = -1$ gives

$$C_1 + C_2 = A.$$

Hence,

$$y_{BL}(\xi) = A + C_2(e^{-a(-1)\xi} - 1). \tag{2.4}$$

The first step in Prandtl's matching principle is to set the two limits

$$\lim_{x\to -1} Y(x) = B\exp\left(\int_{-1}^{1} \frac{b(t)}{a(t)}dt\right) \tag{2.5}$$

and

$$\lim_{\xi\to\infty} y_{BL}(\xi) = A - C_2 \tag{2.6}$$

equal. Thus,

$$C_2 = A - B\exp\left(\int_{-1}^{1} \frac{b(t)}{a(t)}dt\right). \tag{2.7}$$

In (2.5) and (2.6), we have made use of the assumption that $a(x)$ is positive in $[-1, 1]$. The second step in Prandtl's matching principle is to define the uniform approximate solution by

$$y_{\text{unif}}(x) := Y(x) + y_{BL}(\xi) - \text{common part};$$

that is,

$$\begin{aligned} y_{\text{unif}}(x) = {} & B\exp\left(\int_{x}^{1} \frac{b(t)}{a(t)}dt\right) \\ & + A + C_2(e^{-a(-1)(x+1)/\varepsilon} - 1) - A + C_2, \end{aligned}$$

thus obtaining (1.3).

If $a(x) < 0$ in $[-1, 1]$, then the boundary layer region is at the right endpoint $x = 1$. A similar argument will lead to the corresponding formula

$$\begin{aligned} y_{\text{unif}}(x) = {} & A\exp\left(-\int_{-1}^{x} \frac{b(t)}{a(t)}dt\right) \\ & + \left\{B - A\exp\left(-\int_{-1}^{1} \frac{b(t)}{a(t)}dt\right)\right\} e^{a(1)(1-x)/\varepsilon}. \end{aligned} \tag{2.8}$$

A natural question to ask now is "In what sense does $y_{\text{unif}}(x)$ approximate the true solution $y(x)$ of (1.2)?" In many books on applied mathematics (e.g., [2] and [9]), one will find the answer

$$y(x) = y_{\text{unif}}(x) + O(\varepsilon), \tag{2.9}$$

where the O-term is uniform with respect to $x \in [-1, 1]$. Despite its usefulness, equation (2.8) is not entirely correct. For instance, if the boundary value B in (1.2b) is zero, then (1.3) becomes

$$y_{\text{unif}}(x) = Ae^{-a(-1)(1+x)/\varepsilon}, \tag{2.10}$$

which is exponentially small for $x > -1$, and asymptotically zero with respect to the order estimate in (2.9). The more accurate formula is

$$\begin{aligned} y(x) = {} & B \exp\left(\int_x^1 \frac{b(t)}{a(t)} dt\right)[1 + O(\varepsilon)] \\ & + \frac{a(0)}{a(x)}\left\{A - B\exp\left(\int_{-1}^1 \frac{b(t)}{a(t)} dt\right)\right\} \exp\left(\int_{-1}^x \frac{b(t)}{a(t)} dt\right) \\ & \times \exp\left(-\frac{1}{\varepsilon}\int_{-1}^x a(t)dt\right)[1 + O(\varepsilon)]. \end{aligned} \tag{2.11}$$

One can establish this result by using the WKB approximation given in Lecture III; it can also be found in [12]. To illustrate our point, let us consider the simple example

$$\varepsilon y'' + (3 + x)y' + y = 0, \qquad y(-1) = 1, \qquad y(1) = 0. \tag{2.12}$$

Formulas (2.9) and (2.10) give

$$y_{\text{unif}}(x) = e^{-2(1+x)/\varepsilon} + O(\varepsilon). \tag{2.13}$$

In particular, we have

$$y_{\text{unif}}(0) = e^{-2/\varepsilon} + O(\varepsilon). \tag{2.14}$$

But, from (2.11) it follows that

$$y(x) = \frac{3}{2}\exp\left\{-\frac{1}{\varepsilon}\left(\frac{1}{2}x^2 + 3x + \frac{7}{2}\right)\right\}[1 + O(\varepsilon)]. \tag{2.15}$$

Both approximations (2.14) and (2.15) are exponentially small. However, (2.14) only gives

$$y_{\text{unif}}(x) = O(\varepsilon), \tag{2.16}$$

whereas from (2.15) we get

$$y(0) \sim \frac{3}{2}e^{-7/2\varepsilon}. \tag{2.17}$$

3. Internal Layers

We now turn to the case where the coefficient function $a(x)$ in (1.2a) has a zero in the interval $[-1,1]$; that is, condition (1.4) holds. In this case, we shall see that boundary layers may occur in the interior as well as at endpoints. By using matching techniques as in Sec. 1, Bender and Orszag [2, p.458] have constructed a relatively simple asymptotic solution for the boundary-value problem $(1.2a)-(1.2b)$. Indeed, they showed that in (1.4) if $\alpha>0$ and $\beta/\alpha\neq 1,2,3,\cdots$, then

$$\begin{aligned} y_{\text{unif}}(x) = {} & \frac{\Gamma(1-\beta/\alpha)}{\sqrt{2\pi}}(\sqrt{\alpha/\varepsilon})^{\beta/\alpha}e^{-\alpha x^2/4\varepsilon} \\ & \times\left\{A\exp\left(\int_x^{-1}\left[\frac{b(t)}{a(t)}-\frac{\beta}{\alpha t}\right]dt\right)D_{\beta/\alpha-1}(x\sqrt{\alpha/\varepsilon})\right. \\ & \left.+B\exp\left(\int_x^{1}\left[\frac{b(t)}{a(t)}-\frac{\beta}{\alpha t}\right]dt\right)D_{\beta/\alpha-1}(-x\sqrt{\alpha/\varepsilon})\right\}, \end{aligned} \tag{3.1}$$

where $D_\nu(z)$ is the parabolic cylinder function in the notation of Whittaker and Watson. In view of the asymptotic results

$$D_\nu(t)\sim t^\nu e^{-t^2/4}, \qquad D_\nu(-t)\sim\frac{\sqrt{2\pi}}{\Gamma(-\nu)}t^{-\nu-1}e^{t^2/4}, \qquad t\to\infty,$$

and

$$D_\nu(t)\sim\frac{\sqrt{\pi}2^{\nu/2}}{\Gamma\left(\frac{1}{2}-\frac{1}{2}\nu\right)}, \qquad t\to 0,$$

one readily obtains

$$y_{\text{unif}}(x)\sim A\exp\left(-\int_{-1}^{x}\frac{b(t)}{a(t)}dt\right), \qquad \frac{x}{\sqrt{\varepsilon}}\to-\infty, \tag{3.2}$$

$$y_{\text{unif}}(x)\sim B\exp\left(\int_{x}^{1}\frac{b(t)}{a(t)}dt\right), \qquad \frac{x}{\sqrt{\varepsilon}}\to\infty \tag{3.3}$$

and

$$y_{\text{unif}}(x)=O(\varepsilon^{-\beta/2\alpha}), \qquad x=O(\sqrt{\varepsilon}). \tag{3.4}$$

Formulas $(3.2)-(3.4)$ reveal that an *internal boundary layer* occurs when $x=O(\sqrt{\varepsilon})$, i.e., in a neighborhood of $x=0$. A mathmatically rigorous proof of (3.1) has recently been provided by Wong and Yang [18].

For the case $\alpha<0$ and $\beta/\alpha\neq 0,-1,-2,\cdots$, Bender and Orszag [2, p.460] have also given the leading-order uniform asymptotic solution

$$y_{\text{unif}}(x)=Ae^{-a(-1)(x+1)/\varepsilon}+Be^{a(1)(1-x)/\varepsilon} \tag{3.5}$$

to $(1.2a)-(1.2b)$. Although this solution appears to behave like the true solution, we can conclude that this result is not correct on two accounts. First,

it is evident that the approximate solution given in (3.5) depends only on the values of $a(x)$ at the endpoints of $[-1,1]$. This means that the approximation will be the same for any coefficient function $a(x)$ if it takes the same values at the endpoints, which does not seem to be reasonable; see a remark later. Next, by applying (3.5) to the concrete problem

$$\varepsilon y'' - 2xy' + (1+x^2)y = 0, \qquad y(-1) = 2, \qquad y(1) = 1, \tag{3.6}$$

Bender and Orszag gave the asymptotic solution

$$y_{\text{unif}}(x) = 2e^{-2(x+1)/\varepsilon} + e^{-2(1-x)/\varepsilon}. \tag{3.7}$$

The value of $y_{\text{unif}}(x)$ in (3.7) is always positive for $x \in [-1,1]$. This is contrary to the fact that the exact solution is negative for x near the origin; see Figure 9.17 in [2, p.460].

Very recently, a more detailed study of the boundary-value problem (1.2a)–(1.2b) has been made by Wong and Yang [19], in the case when $\alpha < 0$ and $\beta/\alpha \neq 0, -1, -2, \cdots$. They have, in fact, given a rigorous derivation of a uniform asymptotic solution in the whole interval $[-1,1]$. Unlike equation (3.5), their asymptotic formula involves parabolic cylinder functions and the values of the coefficient functions in the entire interval $[-1,1]$. A crucial step in their derivation is to seek an approximate equation for (1.2a). Their uniform asymptotic formula shows that the true solution has boundary layers near the two endpoints $x = -1$ and $x = 1$, and decays exponentially when the independent variable x is away from the boundary layer regions. Comparing their result with Bender and Orszag's formula (3.5), it is found that (3.7) actually gives only the behavior of the exact solution inside the boundary layers. Away from the boundary layers, the approximate solution in (3.7) is incorrect even though it is exponentially decaying.

When the uniform approximation given in [19] is applied to the concrete example (3.6), we get

$$y(0) \sim -\frac{6\sqrt{2\pi}}{\Gamma\left(\frac{1}{4}\right)} e^{1/4} \varepsilon^{-3/4} e^{-1/\varepsilon}, \tag{3.8}$$

whereas Bender and Orszag's result (3.7) gives

$$y_{\text{unif}}(0) = 3e^{-2/\varepsilon}. \tag{3.9}$$

This example again illustrates the fact that the heuristic method of matched asymptotics may lead to incorrect results, when exponentially small terms are involved.

As a follow-up to their first two papers [18] and [19], Wong and Yang [20] also studied the exceptional cases (i) $\beta/\alpha = 1, 2, 3, \cdots$ when $\alpha > 0$ and (ii) $\beta/\alpha = 0, -1, -2, \cdots$ when $\alpha < 0$. Uniform asymptotic solutions have been constructed even in these cases. Analyzing the asymptotic behavior of the

solutions shows that the solutions do not exhibit the bahavior described in their earlier papers. When $\beta/\alpha = 1, 2, 3, \cdots$ and $\alpha > 0$, the solution grows exponentially in a subinterval of $[-1, 1]$; cf. (3.4). When $\beta/\alpha = 0, -1, -2, \cdots$ and $\alpha < 0$, the solution does not vanish exponentially in a subinterval of $[-1, 1]$; cf. (3.5). This phenomenon was first observed by Ackerberg and O'Malley [1], and they called it a "resonance" phenomenon. Summarizing the results of Wong and Yang in their three papers shows that the conditions in the two exceptional cases are necessary and sufficient for resonance.

4. Carrier-Pearson Equation

The power of boundary-layer theory (the method of matched asymptotics) is that it works equally well for nonlinear differential equations. A simple, yet nontrivial, example is the Carrier-Pearson equation [4]

$$\varepsilon u'' + u^2 = 1, \qquad -1 < x < 1, \tag{4.1}$$

with boundary conditions

$$u(-1) = u(1) = 0. \tag{4.2}$$

The outer solutions obtained by setting $\varepsilon = 0$ in (4.1) are $u_{\text{out},-} = -1$ and $u_{\text{out},+} = 1$, neither of which satisfies the boundary conditions at $x = \pm 1$. Therefore, there must exist boundary layers at $x = \pm 1$. Let us examine first the boundary layer at $x = 1$. When we substitute the inner variables

$$\mathrm{X} = \frac{(1-x)}{\delta(\varepsilon)}, \qquad U_{\text{in, right}}(\mathrm{X}) = u(x),$$

into (4.1) and (4.2), we obtain

$$\frac{d^2 U_{\text{in, right}}(\mathrm{X})}{d\mathrm{X}^2} + \frac{\delta^2}{\varepsilon}(U^2_{\text{in, right}} - 1) = 0 \tag{4.3}$$

and

$$U_{\text{in, right}}(0) = 0. \tag{4.4}$$

To balance the order terms in (4.3), we take $\delta(\varepsilon) = \sqrt{\varepsilon}$. From here on we proceed to solve (4.3) and (4.4), and have the solution asymptotically matched with one (or both) of the outer solutions.

We claim that $U_{\text{in, right}}$ cannot approach $u_{\text{out},+}$ as $\mathrm{X} \to \infty$. Suppose so, and let $U_{\text{in, right}}(\mathrm{X}) = 1 + w(\mathrm{X})$. From (4.3) and (4.4), we have

$$w'' + 2w + w^2 = 0 \tag{4.5}$$

and

$$w(0) = -1, \qquad w(\infty) = 0. \tag{4.6}$$

When w tends to zero, equation (4.5) can be approximated by $w'' + 2w = 0$, whose solutions oscillate as $\mathrm{X} \to \infty$ and cannot tend to 0. This confirms our

claim. Thus, we put $U_{\text{in, right}}(\mathrm{X}) = u_{\text{out},-} + w(\mathrm{X}) = -1 + w(\mathrm{X})$. In a similar manner, we have

$$w'' - 2w + w^2 = 0, \tag{4.7}$$

$w(0) = 1$ and $w(\infty) = 0$. Approximating (4.7) by $w'' - 2w = 0$, we anticipate w to decay exponentially as $\mathrm{X} \to \infty$. Hence, we may assume $w'(\infty) = 0$. Multiplying both sides of equation (4.7) by w' and integrating, we obtain

$$(w')^2 - 2w^2 + \frac{2}{3}w^3 = \text{constant}.$$

Since both w' and w tend to zero as $\mathrm{X} \to \infty$, we conclude that the constant is equal to zero and

$$(w')^2 - 2w^2 + \frac{2}{3}w^3 = 0,$$

which is equivalent to the separable first-order equation

$$\frac{dw}{\sqrt{2w^2 - \frac{2}{3}w^3}} = \pm d\mathrm{X}.$$

Integrating this equation gives

$$-\sqrt{2}\tanh^{-1}\sqrt{1 - \frac{1}{3}w} = \pm\mathrm{X} + C.$$

The constant C is determined by the requirement that $w(0) = 1$. Returning to the original variables, we obtain

$$U_{\text{in, right}} = -1 + 3\,\mathrm{sech}^2\left(\pm\frac{1-x}{\sqrt{2\varepsilon}} + \ln(\sqrt{3} + \sqrt{2})\right),$$

which is valid near $x = 1$.

The same argument applies at $x = -1$, and we have

$$U_{\text{in, left}} = -1 + 3\,\mathrm{sech}^2\left(\pm\frac{1+x}{\sqrt{2\varepsilon}} + \ln(\sqrt{3} + \sqrt{2})\right).$$

Matching the outer solution $u_{\text{out},-}$ with $U_{\text{in, right}}$ and $U_{\text{in, left}}$ gives

$$\begin{aligned} u_{\text{unif}}(x) = -1 &+ 3\,\mathrm{sech}^2\left(\pm\frac{1-x}{\sqrt{2\varepsilon}} + \ln(\sqrt{3} + \sqrt{2})\right) \\ &+ 3\,\mathrm{sech}^2\left(\pm\frac{1+x}{\sqrt{2\varepsilon}} + \ln(\sqrt{3} + \sqrt{2})\right) \end{aligned} \tag{4.8}$$

valid over the entire interval $[-1, 1]$.

An alternative, but equivalent, form of (4.8) can also be found in [4]; that is,

$$u_{\text{unif}}(x) = -1 + \frac{12e^{p_1}}{(1 + e^{p_1})^2} + \frac{12e^{p_2}}{(1 + e^{p_2})^2}, \tag{4.9}$$

where

$$p_1 = \pm\sqrt{\frac{2}{\varepsilon}}(1-x) + 2\ln(\sqrt{3}+\sqrt{2}) \tag{4.10}$$

and

$$p_2 = \pm\sqrt{\frac{2}{\varepsilon}}(1+x) + 2\ln(\sqrt{3}+\sqrt{2}). \tag{4.11}$$

Note that the solution in (4.8) is not unique. There are in fact four different solutions, depending on the two choices of plus and minus signs in the boundary layer solutions; see Figure 1.

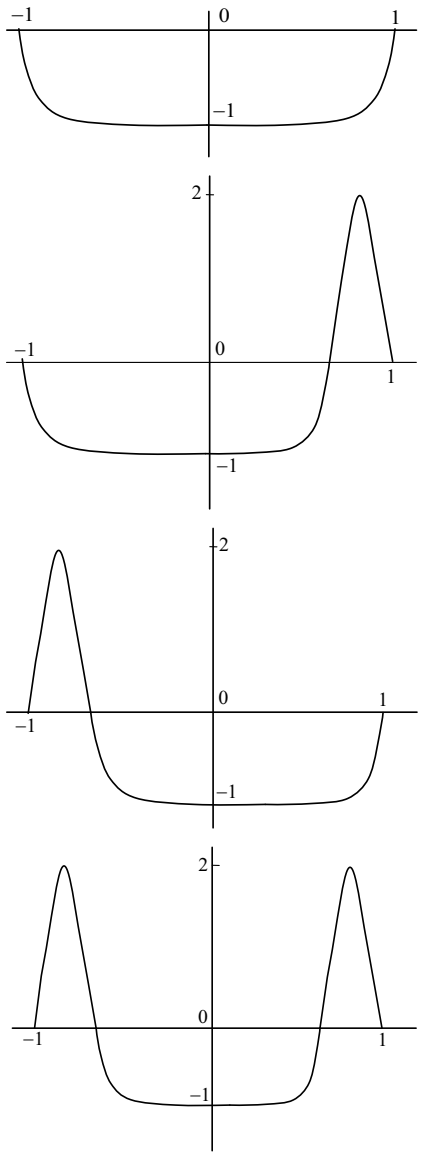

Figure 1. Graph of u_{unif} when $\varepsilon = 0.01$

In view of the non-uniqueness, one may even ask the question: whether a solution of equation (4.1) can have an internal layer, i.e., a narrow region not adjacent to a boundary in which the solution ascends or descends steeply.

Assume that there is an internal layer at $x_0 \in (-1, 1)$, where $1-|x_0| \gg \sqrt{\varepsilon}$. It can be shown, as in (4.3), the thickness of such a layer is $\sqrt{\varepsilon}$. Stretching the variable near $x = x_0$ by the transformation

$$\zeta = \frac{x - x_0}{\sqrt{\varepsilon}}, \qquad q(\zeta) = 1 + u(x),$$

we have

$$q'' - 2q + q^2 = 0, \qquad |\zeta| < \infty, \tag{4.12}$$

with

$$q(-\infty) = q(\infty) = 0. \tag{4.13}$$

Solving (4.12) and (4.13) gives

$$q(\zeta) = \frac{12e^{\sqrt{2}\zeta}}{(1 + e^{\sqrt{2}\zeta})^2} = 3\,\mathrm{sech}^2\left(\frac{\zeta}{\sqrt{2}}\right)$$

or, equivalently,

$$u = -1 + 3\,\mathrm{sech}^2\left(\frac{x - x_0}{\sqrt{2\varepsilon}}\right) \tag{4.14}$$

for x near x_0. The function $q(\zeta)$ takes the value 3 at $\zeta = 0$, decays to zero with exponential rapidity as $|\zeta| \to \infty$; thus, it behaves like a *spike* near x_0 for sufficiently small ε. Matching (4.14) with the outer solution and the two inner solutions near $x = \pm 1$, we get a composite formula

$$\begin{aligned} u_{\mathrm{unif}}(x) = -1 &+ 3\,\mathrm{sech}^2\left(\pm\frac{1-x}{\sqrt{2\varepsilon}} + \ln(\sqrt{3} + \sqrt{2})\right) \\ &+ 3\,\mathrm{sech}^2\left(\pm\frac{1+x}{\sqrt{2\varepsilon}} + \ln(\sqrt{3} + \sqrt{2})\right) \\ &+ 3\,\mathrm{sech}^2\left(\frac{x - x_0}{\sqrt{2\varepsilon}}\right). \end{aligned} \tag{4.15}$$

Formula (4.15) *appears* to be a valid approximation for x in the entire interval. But, by using phase plane analysis, Carrier and Pearson [4, p.204] showed that (4.15) can approximate an exact solution only if

$$x_0 = 0.$$

Thus, for most values of x_0, the solutions given in (4.15) cannot be valid, and they are called *spurious solutions*.

In a recent paper, Ou and Wong [14] have investigated (4.1) – (4.2) from a rigorous point of view. By using a "shooting method", they proved that the formal solutions in (4.15) obtained from the method of matched asymptotics

approximate true solutions with an exponentially small error. The so-called spurious solutions turn out to be approximations of true solutions, when the locations of their "spikes" are properly assigned. They also gave an estimate for the maximum number of spikes that these solutions can have.

As a continuation of [14], Ou and Wong extended their results in [15] to include the singularly perturbed two-point problem

$$\varepsilon u'' + Q(u) = 0, \qquad -1 < x < 1, \tag{4.16}$$

with boundary conditions

$$u(-1) = u(1) = 0 \tag{4.17}$$

or

$$u'(-1) = u'(1) = 0, \tag{4.18}$$

where ε is a small positive parameter. The nonlinear term $Q(u)$ vanishes at $s_-, 0, s_+$ and nowhere else in $[s_-, s_+]$, with $s_- < 0 < s_+$. Furthermore, they assumed that $Q'(s_\pm) < 0$, $Q'(0) > 0$ and

$$\int_{s_-}^{s_+} Q(s)ds = 0. \tag{4.19}$$

Simple examples of functions satisfying these conditions are $Q(u) = u(1-u^2)$ and $Q(u) = \sin \pi u$ for $u \in [-1, 1]$.

Equation (4.16) can be considered as the equation of motion of a nonlinear spring with spring constant large compared to the mass. It is also the steady state version of many partial differential equations arising from physics and biochemistry. Unlike the case $Q(u) = u^2 - 1$, now the solutions exhibit a new phenomenon, known as the *shock* layer, i.e., solutions vary rapidly from one value to another in a very short interval.

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