A Context-free Grammar and a Unification of the Gould Polynomials and the Ramanujan Polynomials

William Y.C. Chen (Joint work with Harold R. L. Yang)
Center for Applied Mathematics, Tianjin University, Tianjin 300072, P. R. China
Email: chenyc@tju.edu.cn

The Gould polynomials are defined by

$$P_n(a,b,c) = c\prod_{i=1}^{n-1}(ia+(n-i)b+c).$$

As a refinement of Cayley's formula, Gessel and Seo showed that $P_n(a,b,c)$ equals the generating function of labeled rooted forests on $[n]$ with respect to certain statistics such as the number of proper vertices. From the symmetry of a and b in the Gould polynomials, one can deduce a symmetry property of the number of proper vertices of labeled rooted forests. Gessel and Seo raised a question of finding a combinatorial proof of this fact. In answer to this question, Hou devised a coding algorithm for labeled rooted forest by means of an insertion procedure.

Another refinement of Cayley's formula is due to Shor, which is based on the notion of improper edges of rooted trees. Zeng established a relation between the number of rooted forests on $[n]$ with k improper edges and polynomials introduced by Ramanujan, called the Ramanujan polynomials.

We consider an enumeration problem on labeled rooted forests. The generating function can be viewed as a unification of the Gould polynomials and the Ramanujan polynomials. We obtain a context-free grammar that gives rise to the refined generating function. This grammar leads to a symmetry property of labeled rooted forests, which can be seen a refinement of the symmetry property of Gessel and Seo.