Comparing the degrees of unconstrained and constrained approximation by polynomials

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It is quite obvious that one should expect that the degree of constrained approximation be not better, in fact, worse than the degree of unconstrained approximation. However, it turns out that in certain cases we can deduce the behavior of the degrees of the former from information about the latter. Surprisingly, this turns out to be the case for the approximation classes A_α, which play an important role in approximation theory.

Let $E_n(f)$ denote the degree of approximation of $f \in C[-1,1]$, by algebraic polynomials of degree $< n$, and assume that we know that for some $\alpha > 0$ and $N \geq 1$,

$$n^\alpha E_n(f) \leq 1, \quad n \geq N.$$

Suppose that $f \in C[-1,1]$, changes its monotonicity or convexity $s \geq 0$ times in $[-1,1]$ ($s = 0$ means that f is monotone or convex, respectively). We are interested in what may be said about its degree of approximation by polynomials of degree $< n$ that are comonotone or coconvex with f. Specifically, if f changes its monotonicity or convexity at $Y_s := \{y_1, \dots, y_s\}$ ($Y_0 = \emptyset$) and the degrees of comonotone and coconvex approximation are denoted by $E_n^{(q)}(f, Y_s)$, $q = 1, 2$, respectively. We investigate when can one say that

$$n^\alpha E_n^{(q)}(f, Y_s) \leq c(\alpha, s, N), \quad n \geq N^*,$$

for some N^*. Clearly, N^*, if it exists at all (we prove it always does), depends on α, s and N. However, it turns out that for certain values of α, s and N, N^* depends also on Y_s, and in some cases even on f itself.