Logarithmic Expansions and the Stability of Periodic Patterns of Localized Spots for Reaction-Diffusion Systems in $\mathbf{R}^2$

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The linear stability of steady-state periodic patterns of localized spots in $\mathbf{R}^2$ for the two-component Gierer-Meinhardt (GM) and Schnakenburg reaction-diffusion models is analyzed in the semi-strong interaction limit corresponding to an asymptotically small diffusion coefficient ϵ^2 of the activator concentration. In the limit $\epsilon \to 0$, localized spots in the activator are centered at the lattice points of a Bravais lattice with constant area $|\Omega|$. To leading order in $\nu = -1/\log \epsilon$, the linearization of the steady-state periodic spot pattern has a zero eigenvalue when the inhibitor diffusivity satisfies $D = D_0/\nu$, for some D_0 independent of the lattice and the Bloch wavevector $\mathbf{k}$. From a combination of the method of matched asymptotic expansions, Floquet-Bloch theory, and the rigorous study of certain nonlocal eigenvalue problems, an explicit analytical formula for the continuous band of spectrum that lies within an $\mathcal{O}(\nu)$ neighborhood of the origin in the spectral plane is derived when $D = D_0/\nu + D_1$, where $D_1 = \mathcal{O}(1)$ is a de-tuning parameter. The periodic pattern is linearly stable when D_1 is chosen small enough so that this continuous band is in the stable left-half plane $\mathrm{Re}(\lambda) < 0$ for all $\mathbf{k}$. Moreover, for both the Schnakenburg and GM models, our analysis identifies a model-dependent objective function, involving the regular part of the Bloch Green's function, that must be maximized in order to determine the specific periodic arrangement of localized spots that constitutes a linearly stable steady-state pattern for the largest value of D. From a numerical computation, based on an Ewald-type algorithm, of the regular part of the Bloch Green's function that defines the objective function, it is shown within the class of oblique Bravais lattices that a regular hexagonal lattice arrangement of spots is optimal for maximizing the stability threshold in D.