On the evaluation of prolate spheroidal wave functions

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Prolate spheroidal wave functions (PSWFs) corresponding to band limit $c > 0$ are the eigenfunctions of the truncated Fourier transform $F_c : L^2[-1,1] \to L^2[-1,1]$ defined via the formula

$$F_c\left[\sigma\right](x) = \int_{-1}^{1} \sigma(t) \cdot e^{icxt}\, dt.$$

In this capacity, PSWFs provide a natural tool for dealing with bandlimited functions defined on an interval, as demonstrated by Slepian et. al. in a sequence of classical papers. (A function $f :\to$ is called bandlimited with band limit $c > 0$ if its Fourier transform is supported on the interval $[-c, c]$.)

Starting with the papers by Slepian et. al., PSWFs have been used as a tool in electrical engineering (design of antenna patterns), digital signal processing (design of digital filters, such as upsampling/downsampling algorithms in acoustics), physics (various wave phenomena, fluid dynamics, uncertainty principles in quantum mechanics), etc.

However, the use of PSWFs has been somewhat crippled by their slightly mysterious reputation as being "difficult to compute". This seems to be related to the fact that the classical ("Bouwkamp") algorithm for their evaluation encounters numerical difficulties for $c > 40$ or so. Moreover, the attempt to diagonalize the operator F_c numerically via straightforward discretization meets with numerical difficulties as well.

In this talk, we describe several numerical algorithms for the evaluation of PSWFs and some associated quantities. While the underlying analysis is somewhat involved, the resulting numerical schemes are quite simple and efficient in practical computations, even for large values of band limit (e.g. $c = 10^6$).